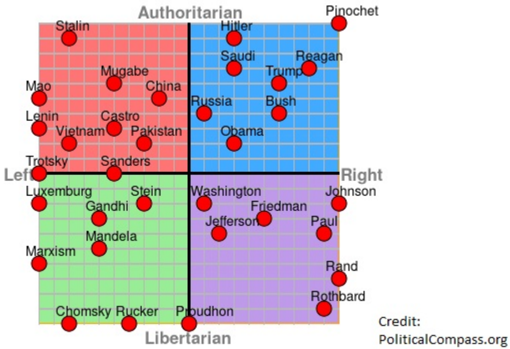
Condorcet Relaxation In Spatial Voting



Arnold Filtser[1] and **Omrit Filtser**[2]

[1]Columbia University [2]Stony Brook University

AAAI 2021

Condorcet criterion

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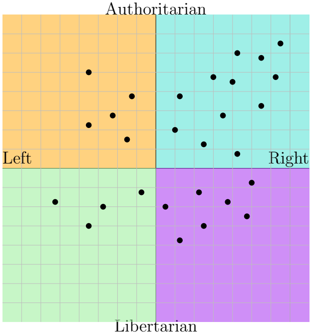

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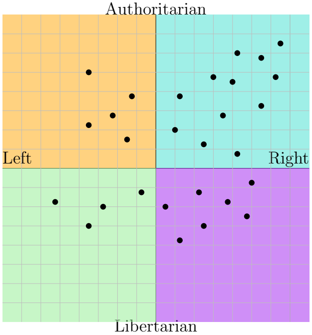

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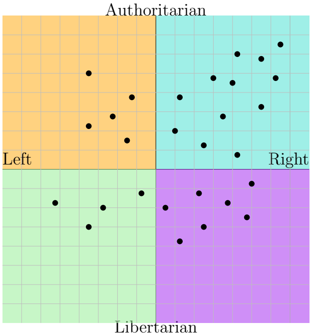

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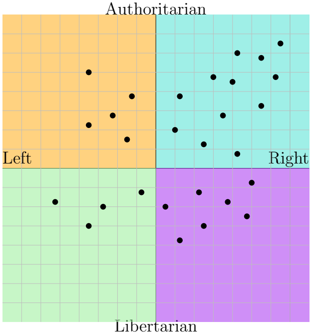

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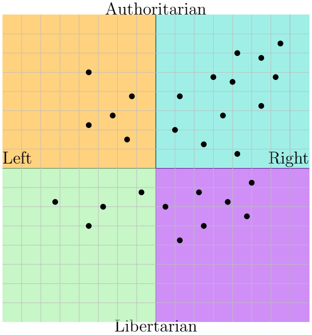

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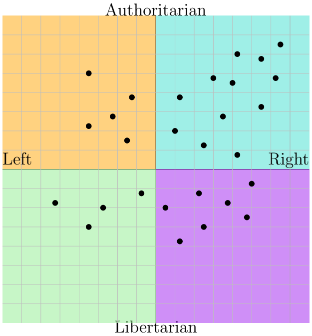

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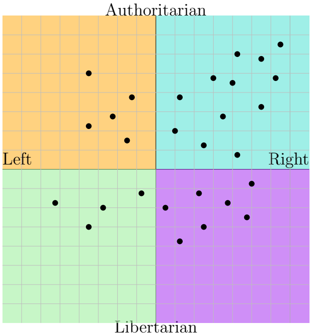

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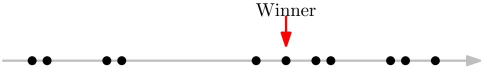

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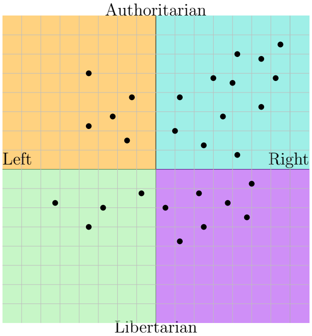

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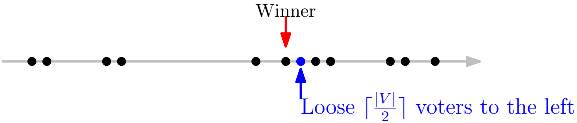

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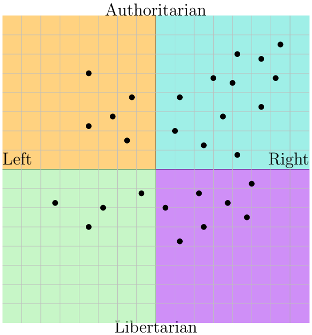

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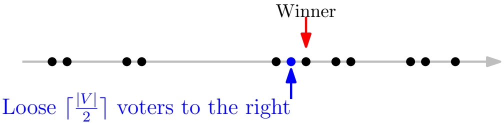

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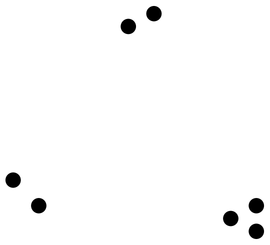

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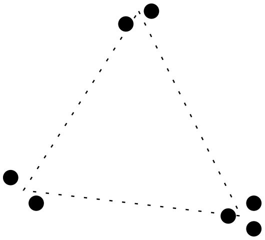

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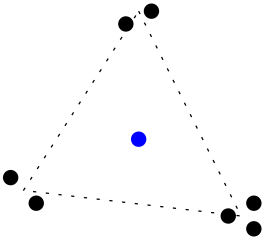

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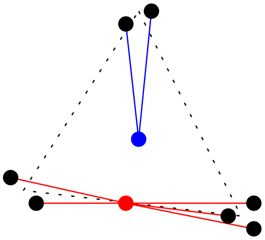

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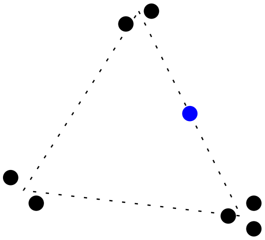

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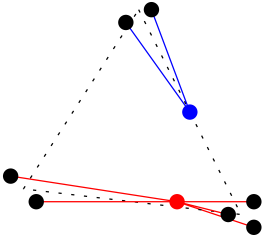

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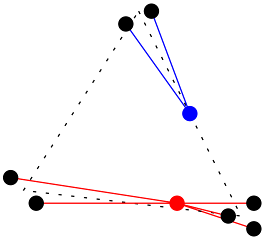

No Condorcet winner!

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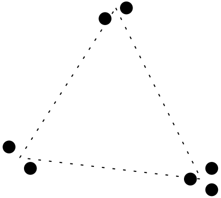

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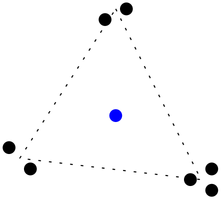

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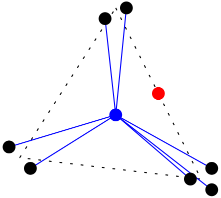

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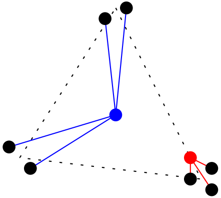

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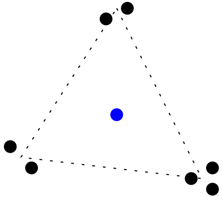

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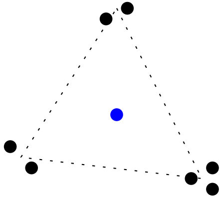

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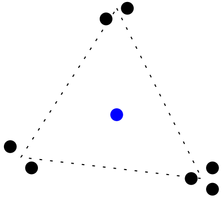

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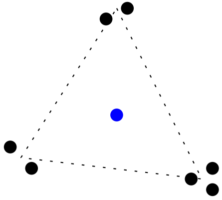

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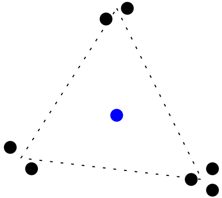

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Given a metric space (X, d), what β should we expect?

What is the **amount of relaxation** needed in order to reach a stable decision **for any set of voters** V in X?

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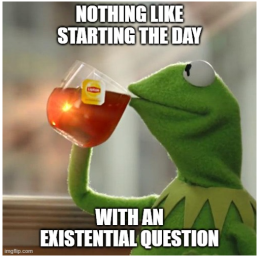

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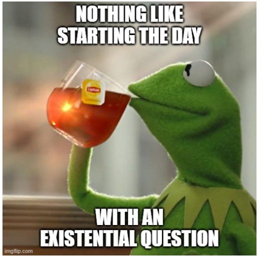

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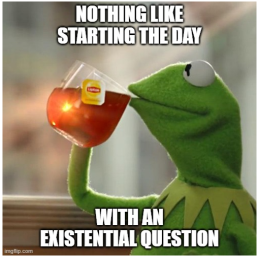

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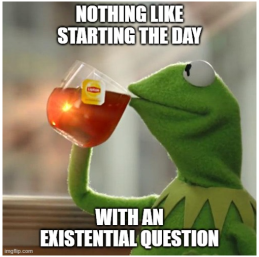

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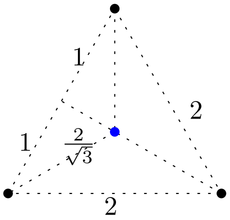

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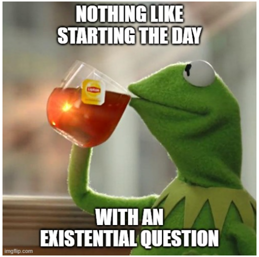

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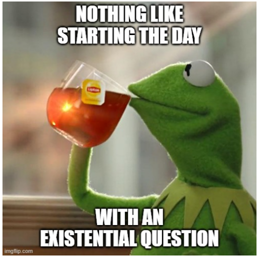

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- $\beta^*_{(\mathbb{R}^d,\|\cdot\|_2)} \geq \frac{1}{\sqrt{d}}$: For every V in $\mathbb{R}^d$, there exists a $\frac{1}{\sqrt{d}}$-plurality point.

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Aronov et al. (2020):

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- Actually, for every metric space: $\beta^*_{(X,d)} \geq \sqrt{2} - 1$

Results

Aronov et al. (2020):

- $\beta^*_{(\mathbb{R}^2,\|\cdot\|_2)} = \frac{\sqrt{3}}{2}$
- $\beta^*_{(\mathbb{R}^d,\|\cdot\|_2)} \in [\frac{1}{\sqrt{d}}, \frac{\sqrt{3}}{2}]$

Our results:

- Spatial voting: $\beta^*_{(\mathbb{R}^d,\|\cdot\|_2)} > 0.557$ $\rightarrow$ constant!

  For $d \geq 4$: $0.577 > \frac{1}{\sqrt{d}} \Rightarrow \beta^*_{(\mathbb{R}^d,\|\cdot\|_2)} \in (0.557, \frac{\sqrt{3}}{2}]$
- Actually, for <u>every metric space</u>: $\beta^*_{(X,d)} \geq \sqrt{2} - 1$
- Also, there exist a metric space with $\beta^*_{(X,d)} \leq \frac{1}{2}$

General metric spaces

How small can $\beta^*_{(X,d)}$ be?

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$$\Rightarrow \beta^* \in [\sqrt{2}-1, \tfrac{1}{2}]$$

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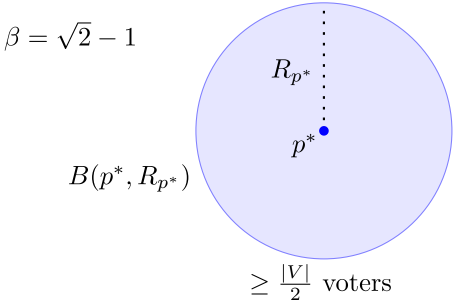

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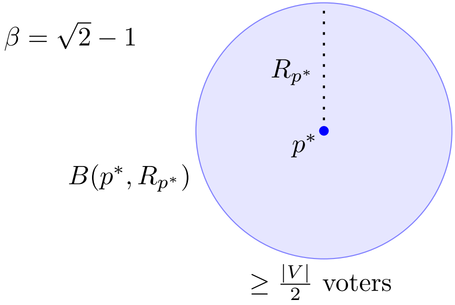

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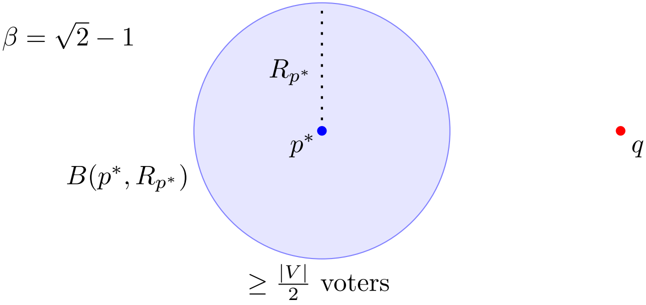

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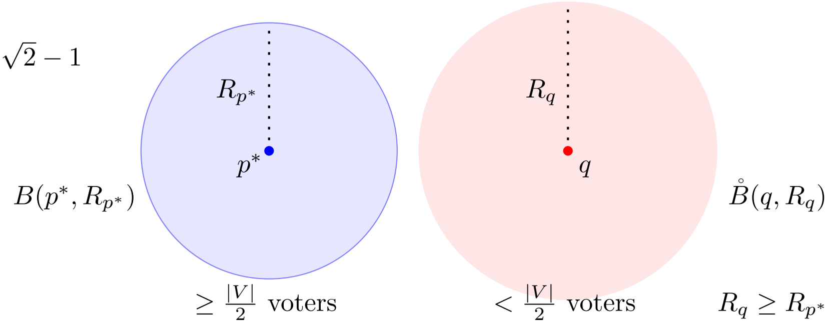
$\beta = \sqrt{2}-1$



$R_q \geq R_{p^*}$

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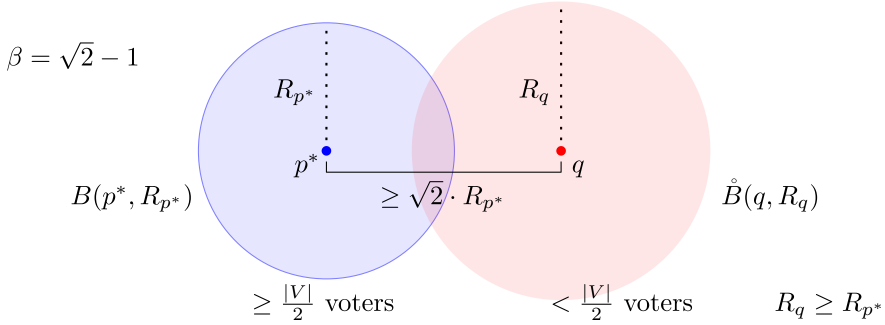

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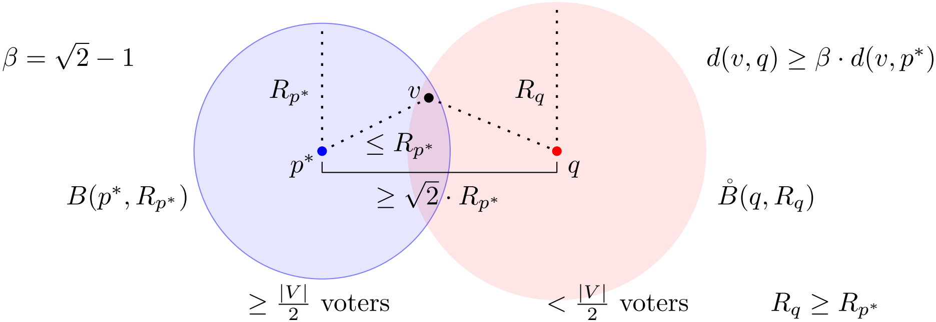

$\boxed{d(p^*, q) \geq \sqrt{2} \cdot R_{p^*} \quad \Rightarrow \quad \text{every } v \in B(p^*, R_{p^*})\ \beta\text{-prefer } p^* \text{ over } q.}$

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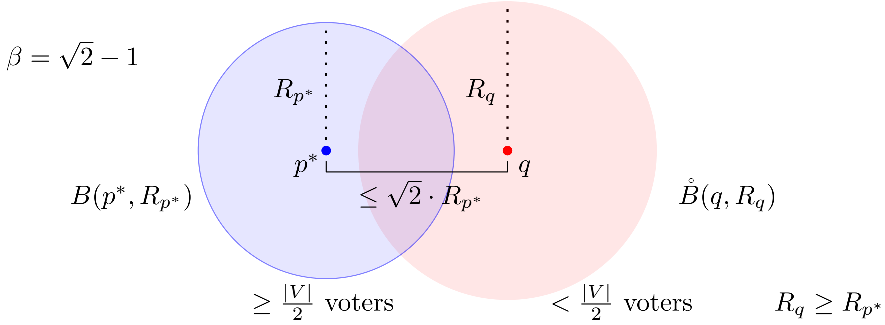

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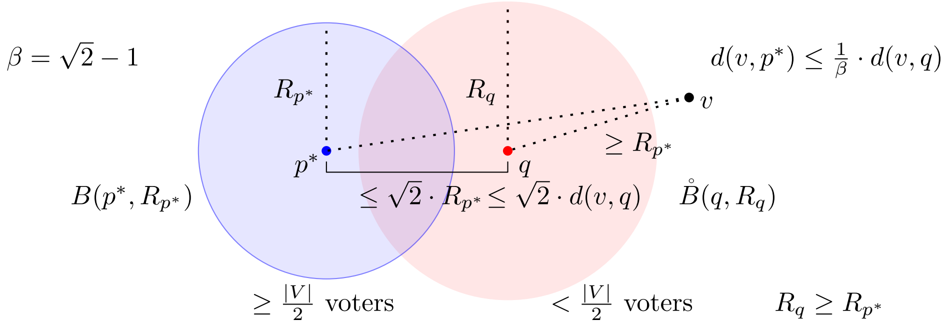

$$\boxed{d(p^*, q) \leq \sqrt{2} \cdot R_{p^*} \quad \Rightarrow \quad \text{every } v \notin \mathring{B}(q, R_q)\ \beta\text{-prefer } p^* \text{ over } q.}$$

Theorem 2: β^* is at most $\frac{1}{2}$

There exists (X, d) and V in X, s.t. there is no β-plurality point for any $\beta > \frac{1}{2}$

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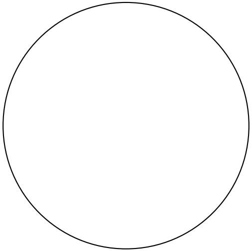

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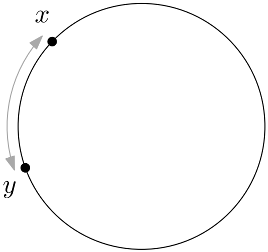

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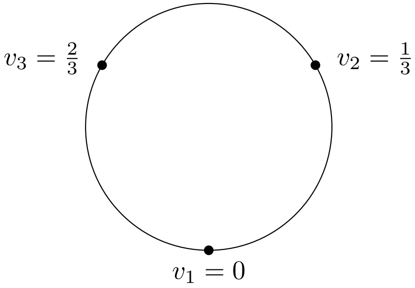

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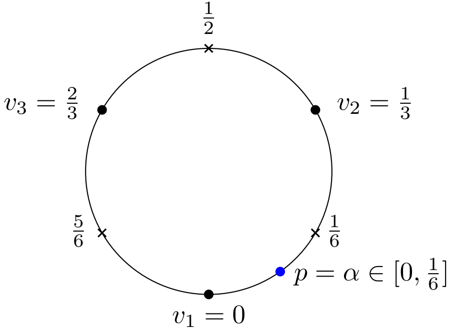

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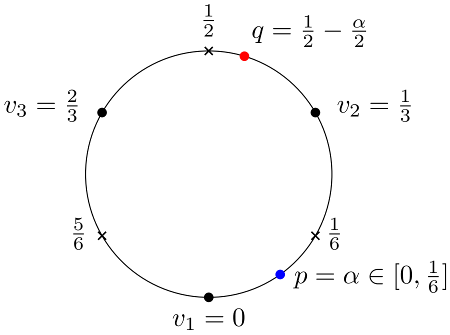

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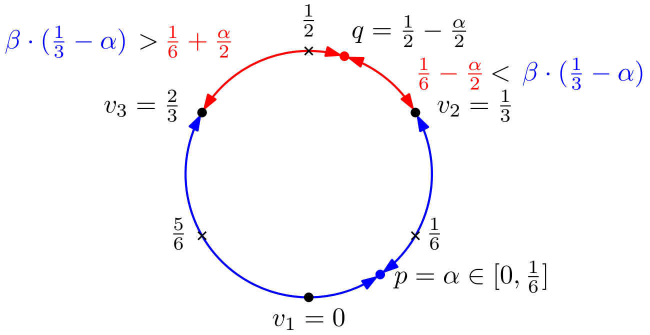

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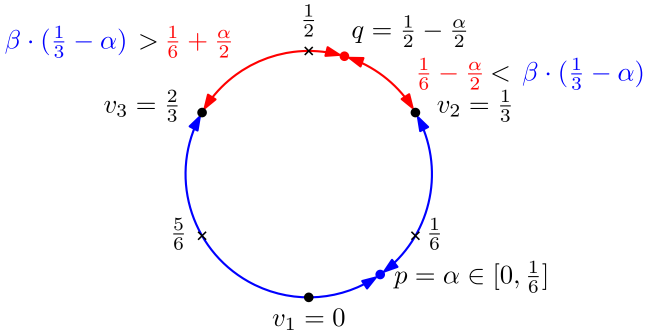

*Actually, for this metric space $\beta^*_{(X,d)} = \frac{1}{2}$

Conclusion and open questions

We show:

- $\beta^* \in [\sqrt{2} - 1, \frac{1}{2}]$
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Main open question:
closing these two gaps.

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Conjecture:

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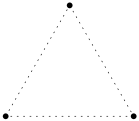
Why? The equilateral triangle is probably the worst case example.

A plurality point must "win" $\frac{2}{3}$ of the votes:



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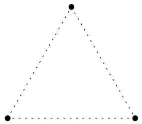

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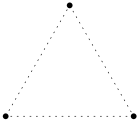

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Thank You!