Condorcet Relaxation In Spatial Voting

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Condorcet criterion

Voters $V$ — multiset in a metric space $(X, d)$

Goal: reach joint a decision — a point in $X$.

Rule: $v \in V$ “prefer” $p$ over $q$ if $d(p, v) \leq d(q, v)$
Condorcet criterion

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**Condorcet winner:** a point in $X$ who would win a two-candidate election against any other point in a plurality vote.
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  - Black 1948
  - Downs 1957
  - Plott 1967
  - Enelow and Hinich 1983

Example: \(d = 1\) (points on the line) \(\rightarrow\) median always wins!
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**Example:** $d = 1$ (points on the line) $\rightarrow$ median always wins!
Fact: For $d > 1$, a Condorcet winner rarely exists...
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No Condorcet winner!
Relaxing the Condorcet criterion

$\beta$ - relaxation parameter

Goal: reach a joint decision — a point in $X$.

Rule: $v \in V$ "$\beta$-prefer" $p$ over $q$ if $\beta \cdot d(p, v) \leq d(q, v)$.

$\beta$-plurality point for $V$: $p \in X$ s.t. $\forall q \in X$, at least $|V|/2$ voters "$\beta$-prefer" $p$ over $q$.

[Aronov, de Berg, Gudmundsson, and Horton, SoCG'20]

$\beta = 1 \iff$ a Condorcet winner

Example: $\beta = 5/12$
Relaxing the Condorcet criterion

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\[ \beta = 1 \iff \text{a Condorcet winner} \]
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\[ \beta = 1 \iff \text{a Condorcet winner} \]

**Example**: $\beta = \frac{1}{2}$
Relaxing the Condorcet criterion

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![Diagram](image-url)
Relaxing the Condorcet criterion

- relaxation parameter

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**β** - relaxation parameter

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**β**-plurality point for \( V \):

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\[ \beta = 1 \iff \text{a Condorcet winner} \]

**Example:** \( \beta = \frac{1}{2} \)
The algorithmic question

What is the \textbf{largest} $\beta$ such that $V$ admits a $\beta$-plurality point?

Aronov et al. (2020): EPTAS for computing $\beta(R^d, \|\cdot\|_2)(V)$

$\beta$ becomes larger $\Rightarrow$ we are “closer” to having a Condorcet winner

Given a metric space $(X,d)$, what $\beta$ should we expect?

What is the amount of relaxation needed in order to reach a stable decision for any set of voters $V$ in $X$?
The algorithmic question

What is the largest $\beta$ such that $V$ admits a $\beta$-plurality point?

$$\beta_{(x,d)}(V) = \sup\{\beta \mid V \text{ admits a } \beta\text{-plurality point}\}$$
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Given a metric space $(X,d)$, what $\beta$ should we expect?

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The existential question

Given a metric space $(X, d)$:
What is the largest $\beta$ s.t.
 every multiset $V$ in $X$ admits a $\beta$-plurality point?

$\beta^* (X, d) = \sup \{ \beta \mid \text{every finite multiset } V \text{ in } X \text{ admits a } \beta\text{-plurality point} \}$

Aronov et al. (2020):
$\beta^* (\mathbb{R}^2, \| \cdot \|_2) \leq \sqrt{\frac{7}{2}}$
The existential question

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What is the largest $\beta$ s.t.

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Aronov et al. (2020):

$\beta^*(\mathbb{R}^2, \|\cdot\|_2) \leq \sqrt{\frac{3}{2}}$: When $V$ is an equilateral triangle,

$\beta(\mathbb{R}^2, \|\cdot\|_2)(V) \leq \sqrt{\frac{3}{2}}$. 

\[\frac{7}{12}\]
The existential question

Given a metric space \((X,d)\):
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Aronov et al. (2020):
\(\beta^*(\mathbb{R}^2,\|\cdot\|_2) \leq \frac{\sqrt{3}}{2}\)
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Given a metric space \((X, d)\):
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Aronov et al. (2020):

\(\beta^*_2 \leq \frac{\sqrt{3}}{2}\): When \(V\) is an equilateral triangle, \(\beta_2(V) \leq \frac{\sqrt{3}}{2}\)
The existential question

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\beta^*(\mathbb{R}^2, \|\cdot\|_2) \leq \frac{\sqrt{3}}{2}: \text{When } V \text{ is an equilateral triangle, } \beta^*(\mathbb{R}^2, \|\cdot\|_2)(V) \leq \frac{\sqrt{3}}{2}
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\beta^*(\mathbb{R}^2, \|\cdot\|_2) \geq \frac{\sqrt{3}}{2}: \text{For every } V \text{ in } \mathbb{R}^2, \text{ there exists a } \frac{\sqrt{3}}{2}\text{-plurality point.}
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Given a metric space \((X, d)\):
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**Aronov et al. (2020):**

- \(\beta^*_{(\mathbb{R}^2, \|\cdot\|_2)} \leq \frac{\sqrt{3}}{2}\): When \(V\) is an equilateral triangle, \(\beta_{(\mathbb{R}^2, \|\cdot\|_2)}(V) \leq \frac{\sqrt{3}}{2}\)

- \(\beta^*_{(\mathbb{R}^2, \|\cdot\|_2)} \geq \frac{\sqrt{3}}{2}\): For every \(V\) in \(\mathbb{R}^2\), there exists a \(\frac{\sqrt{3}}{2}\)-plurality point.

- \(\beta^*_{(\mathbb{R}^d, \|\cdot\|_2)} \geq \frac{1}{\sqrt{d}}\): For every \(V\) in \(\mathbb{R}^d\), there exists a \(\frac{1}{\sqrt{d}}\)-plurality point.
Results

Aronov et al. (2020):

- $\beta^*_{(\mathbb{R}^2, \| \cdot \|_2)} = \frac{\sqrt{3}}{2}$

- $\beta^*_{(\mathbb{R}^d, \| \cdot \|_2)} \in \left[ \frac{1}{\sqrt{d}}, \frac{\sqrt{3}}{2} \right]$
Aronov et al. (2020):

\[ \beta^{*}_{(\mathbb{R}^{2}, \|\cdot\|_{2})} = \frac{\sqrt{3}}{2} \]

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Our results:

- Spatial voting: \[ \beta^{*}_{(\mathbb{R}^{d}, \|\cdot\|_{2})} > 0.557 \rightarrow \text{constant!} \]
Results

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- $\beta^*_{(\mathbb{R}^2, \|\cdot\|_2)} = \frac{\sqrt{3}}{2}$
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- Spatial voting: $\beta^*_{(\mathbb{R}^d, \|\cdot\|_2)} > 0.557 \rightarrow \text{constant!}$
  
  For $d \geq 4$: $0.577 > \frac{1}{\sqrt{d}} \Rightarrow \beta^*_{(\mathbb{R}^d, \|\cdot\|_2)} \in (0.557, \frac{\sqrt{3}}{2}]$
Results

Aronov et al. (2020):

- $\beta^*(\mathbb{R}^2, \|\cdot\|_2) = \frac{\sqrt{3}}{2}$
- $\beta^*(\mathbb{R}^d, \|\cdot\|_2) \in \left[\frac{1}{\sqrt{d}}, \frac{\sqrt{3}}{2}\right]$

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- For $d \geq 4$: $0.577 > \frac{1}{\sqrt{d}} \Rightarrow \beta^*(\mathbb{R}^d, \|\cdot\|_2) \in (0.557, \frac{\sqrt{3}}{2}]$
- Actually, for every metric space: $\beta^*(\mathcal{X}, d) \geq \sqrt{2} - 1$
Results

Aronov et al. (2020):

- $\beta^*_{(\mathbb{R}^2, \|\cdot\|_2)} = \frac{\sqrt{3}}{2}$
- $\beta^*_{(\mathbb{R}^d, \|\cdot\|_2)} \in \left[ \frac{1}{\sqrt{d}}, \frac{\sqrt{3}}{2} \right]$

Our results:

- Spatial voting: $\beta^*_{(\mathbb{R}^d, \|\cdot\|_2)} > 0.557 \rightarrow \text{constant!}$

  For $d \geq 4$: $0.577 > \frac{1}{\sqrt{d}} \Rightarrow \beta^*_{(\mathbb{R}^d, \|\cdot\|_2)} \in (0.557, \frac{\sqrt{3}}{2} ]$

- Actually, for every metric space: $\beta^*_{(X, d)} \geq \sqrt{2} - 1$

- Also, there exist a metric space with $\beta^*_{(X, d)} \leq \frac{1}{2}$
General metric spaces

How small can $\beta^*(X, d)$ be?

$$\beta^* = \inf \left\{ \beta^*(X, d) \mid (X, d) \text{ is a metric space} \right\}$$
General metric spaces

How small can $\beta^{*}(X,d)$ be?

$$\beta^{*} = \inf \left\{ \beta^{*}(X,d) \mid (X,d) \text{ is a metric space} \right\}$$

**Theorem 1:** $\beta^{*}$ is at least $\sqrt{2} - 1$
General metric spaces

How small can $\beta^*(X,d)$ be?

$$\beta^* = \inf \left\{ \beta^*(X,d) \mid (X,d) \text{ is a metric space} \right\}$$

Theorem 1: $\beta^*$ is at least $\sqrt{2} - 1$

Theorem 2: $\beta^*$ is at most $\frac{1}{2}$
General metric spaces

How small can $\beta^*(X,d)$ be?

$$\beta^* = \inf \left\{ \beta^*(X,d) \mid (X,d) \text{ is a metric space} \right\}$$

**Theorem 1:** $\beta^*$ is at least $\sqrt{2} - 1$

**Theorem 2:** $\beta^*$ is at most $\frac{1}{2}$

$$\Rightarrow \beta^* \in [\sqrt{2} - 1, \frac{1}{2}]$$
Theorem 1: \( \beta^* \) is at least \( \sqrt{2} - 1 \)

For every \((X, d)\) and \(V\) in \(X\), there exists a \( \sqrt{2} - 1 \)-plurality point.
Theorem 1: $\beta^*$ is at least $\sqrt{2} - 1$

For every $(X, d)$ and $V$ in $X$, there exists a $\sqrt{2} - 1$-plurality point.

$p$ - some point in $X$  \hspace{1cm} $R_p$ - minimum radius s.t. $B(p, R_p)$ contains $\geq \frac{|V|}{2}$ voters.
Theorem 1: $\beta^*$ is at least $\sqrt{2} - 1$

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$p^*$ - point in $X$ with the smallest $R_{p^*}$

$$\beta = \sqrt{2} - 1$$

$$B(p^*, R_{p^*}) \geq \frac{|V|}{2} \text{ voters}$$
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**Claim:** $p^*$ is a $\sqrt{2} - 1$-plurality point.

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\beta = \sqrt{2} - 1
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Claim: $p^*$ is a $\sqrt{2} - 1$-plurality point.

$\beta = \sqrt{2} - 1$

$B(p^*, R_p^*)$ contains $\geq \frac{|V|}{2}$ voters

$\hat{B}(q, R_q)$ contains $< \frac{|V|}{2}$ voters

$R_q \geq R_p^*$
Theorem 1: $\beta^*$ is at least $\sqrt{2} - 1$

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Theorem 1: $\beta^*$ is at least $\sqrt{2} - 1$

For every $(X, d)$ and $V$ in $X$, there exists a $\sqrt{2} - 1$-plurality point.

$p$ - some point in $X$  
$p^*$ - point in $X$ with the smallest $R_p^*$  

Claim: $p^*$ is a $\sqrt{2} - 1$-plurality point.

\[ \beta = \sqrt{2} - 1 \]

\[ d(v, q) \geq \beta \cdot d(v, p^*) \]

\[ R_q \geq R_p^* \]

\[ d(p^*, q) \geq \sqrt{2} \cdot R_p^* \Rightarrow \text{every } v \in B(p^*, R_p^*) \beta\text{-prefer } p^* \text{ over } q. \]
Theorem 1: $\beta^*$ is at least $\sqrt{2} - 1$

For every $(X, d)$ and $V$ in $X$, there exists a $\sqrt{2} - 1$-plurality point.

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Claim: $p^*$ is a $\sqrt{2} - 1$-plurality point.
Theorem 2: $\beta^* \leq \frac{1}{2}$

There exists $(X, d)$ and $V$ in $X$, s.t. there is no $\beta$-plurality point for any $\beta > \frac{1}{2}$
Theorem 2: $\beta^*$ is at most $\frac{1}{2}$

There exists $(X, d)$ and $V$ in $X$, s.t. there is no $\beta$-plurality point for any $\beta > \frac{1}{2}$

**Metric space:** $C$ cycle of length 1, shortest path distance.
Theorem 2: $\beta^*$ is at most $\frac{1}{2}$

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Theorem 2: $\beta^*$ is at most $\frac{1}{2}$

There exists $(X, d)$ and $V$ in $X$, s.t. there is no $\beta$-plurality point for any $\beta > \frac{1}{2}$

**Metric space:** $C$ cycle of length 1, shortest path distance.

Assume $\beta > \frac{1}{2}$
Theorem 2: $\beta^*$ is at most $\frac{1}{2}$

There exists $(X, d)$ and $V$ in $X$, s.t. there is no $\beta$-plurality point for any $\beta > \frac{1}{2}$

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**Metric space:** $C$ cycle of length 1, shortest path distance.

Assume $\beta > \frac{1}{2}$

\[
\begin{align*}
    v_1 &= 0 \\
    v_2 &= \frac{1}{3} \\
    v_3 &= \frac{2}{3} \\
    p &= \alpha \in [0, \frac{1}{6}] \\
    q &= \frac{1}{2} - \frac{\alpha}{2}
\end{align*}
\]
Theorem 2: $\beta^*$ is at most $\frac{1}{2}$

There exists $(X, d)$ and $V$ in $X$, s.t. there is no $\beta$-plurality point for any $\beta > \frac{1}{2}$

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Assume $\beta > \frac{1}{2}$

\[
\beta \cdot \left(\frac{1}{3} - \alpha\right) > \frac{1}{6} + \frac{\alpha}{2}
\]

\[
\frac{1}{6} - \frac{\alpha}{2} < \beta \cdot \left(\frac{1}{3} - \alpha\right)
\]

\[
p = \alpha \in [0, \frac{1}{6}]
\]

\[
v_1 = 0
\]

\[
v_2 = \frac{1}{3}
\]

\[
v_3 = \frac{2}{3}
\]

\[
q = \frac{1}{2} - \frac{\alpha}{2}
\]

\[
\frac{5}{6}
\]

\[
\frac{1}{6}
\]
Theorem 2: $\beta^*$ is at most $\frac{1}{2}$

There exists $(X, d)$ and $V$ in $X$, s.t. there is no $\beta$-plurality point for any $\beta > \frac{1}{2}$

**Metric space:** $C$ cycle of length 1, shortest path distance.

Assume $\beta > \frac{1}{2}$

$$\beta \cdot \left(\frac{1}{3} - \alpha\right) > \frac{1}{6} + \frac{\alpha}{2}$$

$\frac{1}{2} = q = \frac{1}{2} - \frac{\alpha}{2}$

$\frac{1}{6} - \frac{\alpha}{2} < \beta \cdot \left(\frac{1}{3} - \alpha\right)$

*Actually, for this metric space $\beta^*_{(X,d)} = \frac{1}{2}$*
Conclusion and open questions

We show:

- $\beta^* \in [\sqrt{2} - 1, \frac{1}{2}]$
- $\beta^*(\mathbb{R}^d, \|\cdot\|_2) \in (0.557, \frac{\sqrt{3}}{2}]$

Main open question:

closing these two gaps.

Why?
The equilateral triangle is probably the worst case example. A plurality point must "win" $\frac{2}{3}$ of the votes.

Conclusion: If indeed $\beta^*(\mathbb{R}^d, \|\cdot\|_2) = \frac{\sqrt{3}}{2}$, then the amount of "compromise" that we need to make in order to find a "winner" is relatively small.
Conclusion and open questions

We show:

- $\beta^* \in [\sqrt{2} - 1, \frac{1}{2}]$
- $\beta^*(\mathbb{R}^d, \|\cdot\|_2) \in (0.557, \frac{\sqrt{3}}{2}]$

Main open question:

closing these two gaps.

Conjecture:

- $\beta^* = \frac{1}{2}$
- $\beta^*(\mathbb{R}^d, \|\cdot\|_2) = \frac{\sqrt{3}}{2}$ for $d \geq 2$

Why? The equilateral triangle is probably the worst case example.

A plurality point must "win" $\frac{2}{3}$ of the votes:
Conclusion and open questions

We show:

▶ $\beta^* \in [\sqrt{2} - 1, \frac{1}{2}]$

▶ $\beta^*_{(\mathbb{R}^d, \|\cdot\|_2)} \in (0.557, \frac{\sqrt{3}}{2}]$

Main open question:

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Conjecture:

▶ $\beta^* = \frac{1}{2}$

▶ $\beta^*_{(\mathbb{R}^d, \|\cdot\|_2)} = \frac{\sqrt{3}}{2}$ for $d \geq 2$

Why? The equilateral triangle is probably the worst case example.

A plurality point must “win” $\frac{2}{3}$ of the votes:

Conclusion: If indeed $\beta^*_{(\mathbb{R}^d, \|\cdot\|_2)} = \frac{\sqrt{3}}{2} \approx 0.866$ then the amount of “compromise” that we need to make in order to find a “winner” is relatively small.
Conclusion and open questions

We show:

- $\beta^* \in [\sqrt{2} - 1, \frac{1}{2}]$
- $\beta^*_{(\mathbb{R}^d, \|\cdot\|_2)} \in (0.557, \frac{\sqrt{3}}{2}]$

Main open question: closing these two gaps.

Conjecture:

- $\beta^* = \frac{1}{2}$
- $\beta^*_{(\mathbb{R}^d, \|\cdot\|_2)} = \frac{\sqrt{3}}{2}$ for $d \geq 2$

Why? The equilateral triangle is probably the worst case example.

A plurality point must "win" $\frac{2}{3}$ of the votes:

Conclusion: If indeed $\beta^*_{(\mathbb{R}^d, \|\cdot\|_2)} = \frac{\sqrt{3}}{2} \approx 0.866$ then the amount of "compromise" that we need to make in order to find a "finder" is relatively small.

Thank You!