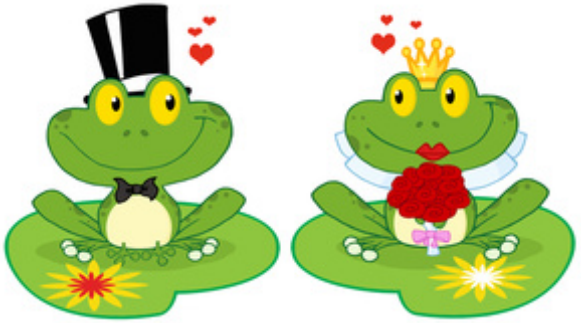
Algorithms for the discrete Fréchet distance under translation

by Omrit Filtser

Ben-Gurion University of the Negev

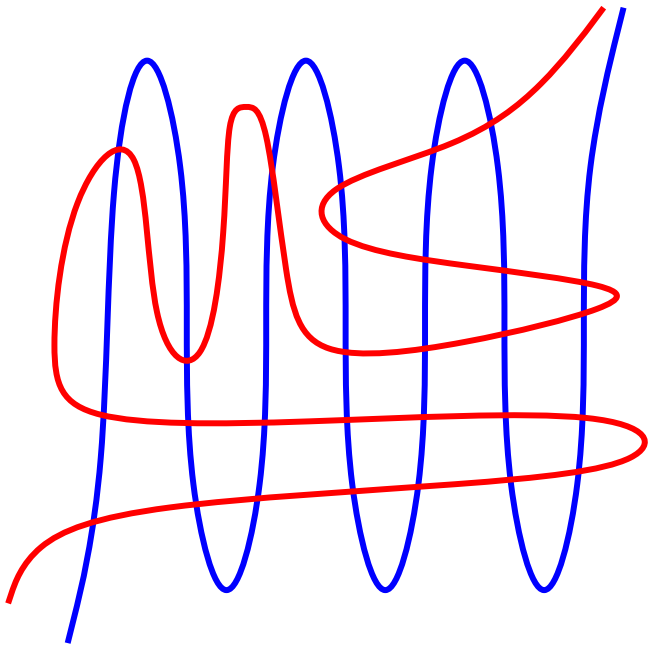


December 27, 2017

Based on joint work with Matthew J. Katz

Similarity of curevs

To what extent the two given curves resemble each other?



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Applications: Signature verification

Similarity of curevs

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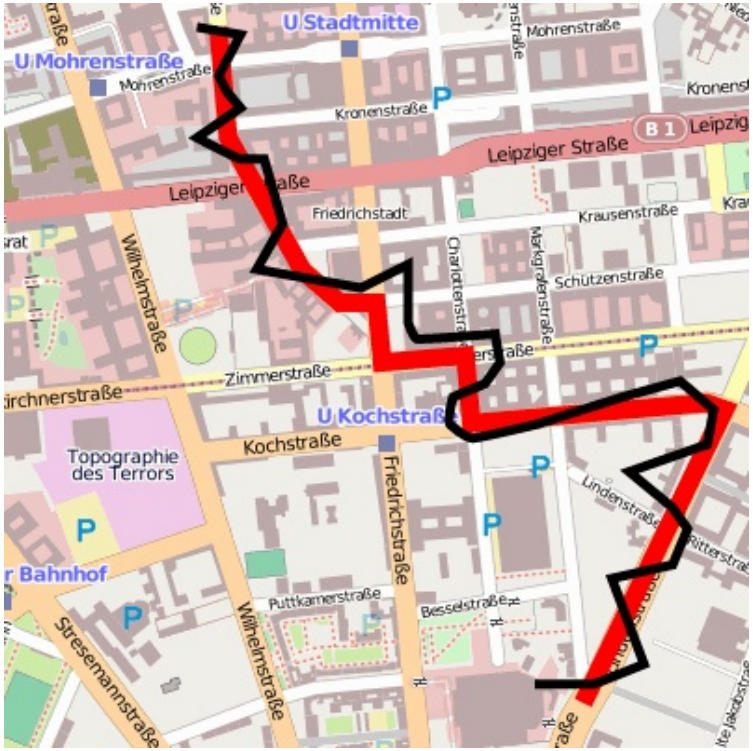
Applications: Map-matching of vehicle tracking data



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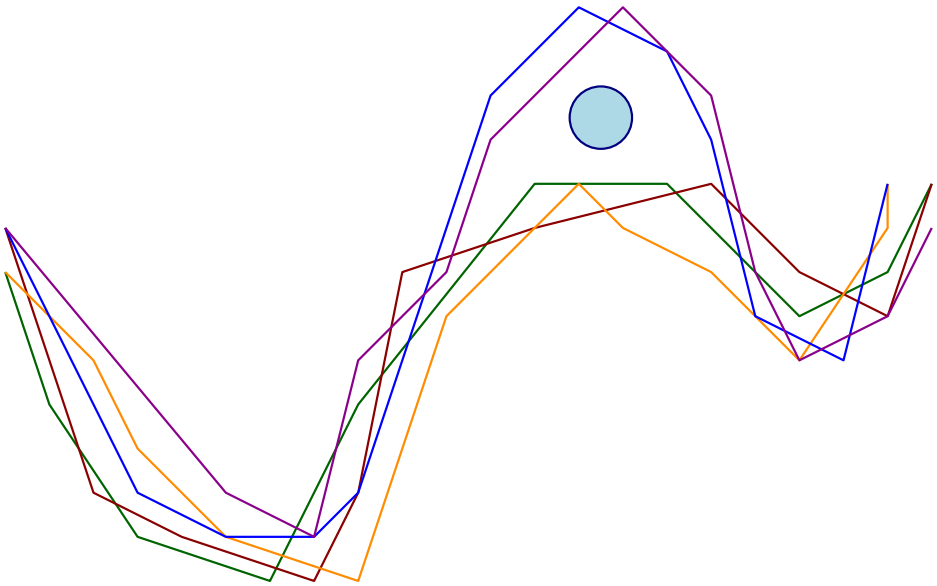
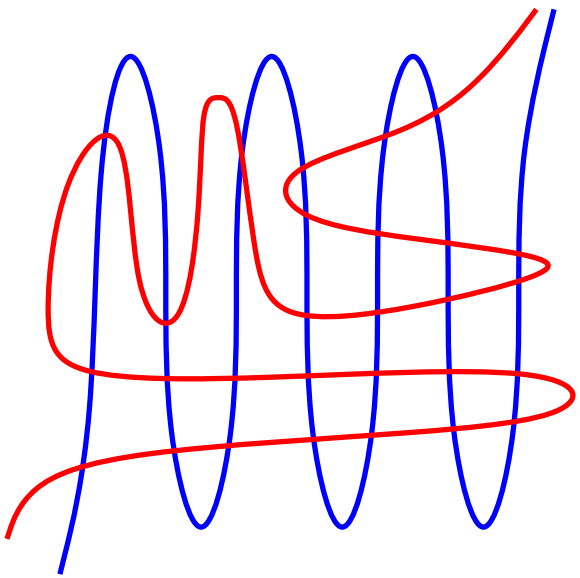
Applications: Analysis of moving objects



How to compare curves?

What distance measure between curves should be used?



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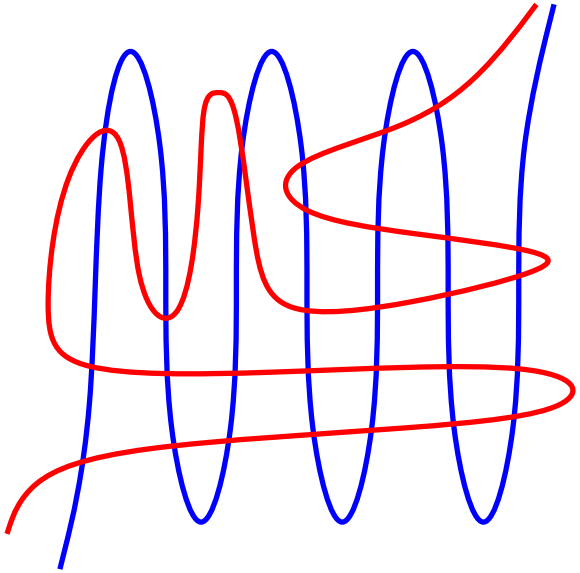

- Hausdorff distance.
  $d_H = \max\{\sup_{a \in A} \inf_{b \in B} d(a,b), \sup_{b \in B} \inf_{a \in A} d(a,b)\}$

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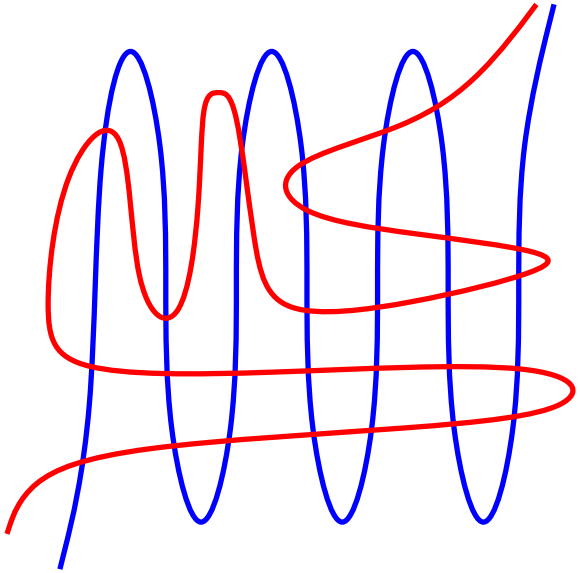

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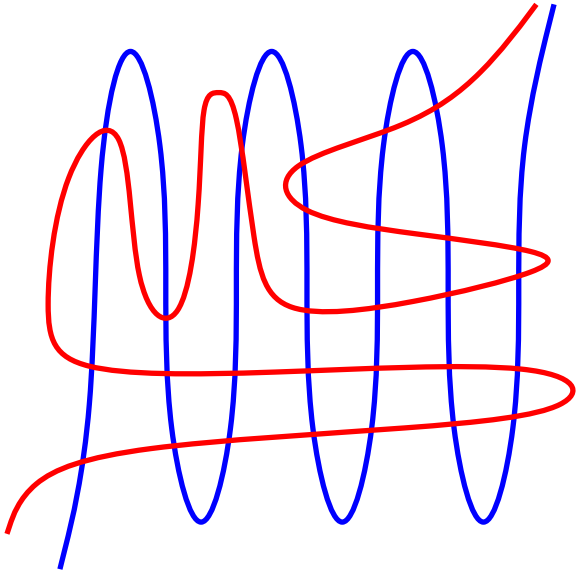

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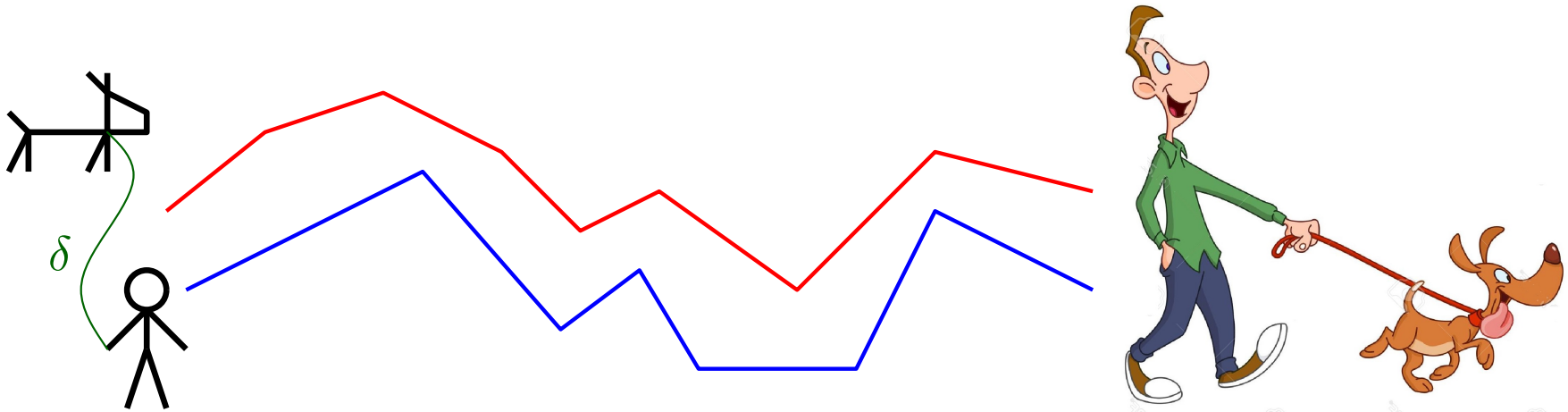
✓ Fréchet distance.

The (continuous) Fréchet distance

A person and a dog connected by a leash of length δ.
They walk along the curves A and B, respectively, no backtracking.



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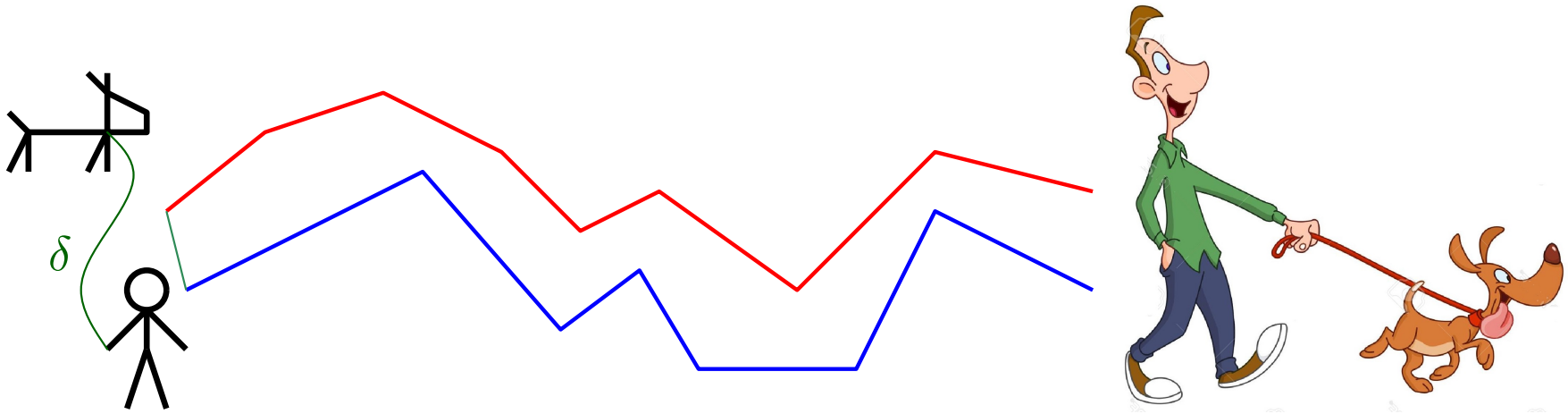

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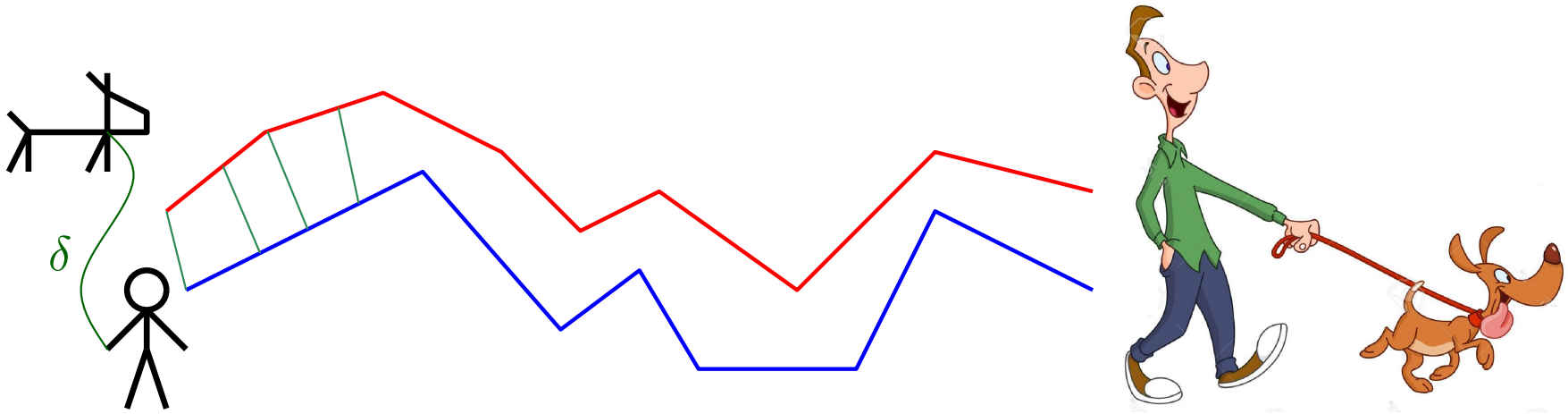

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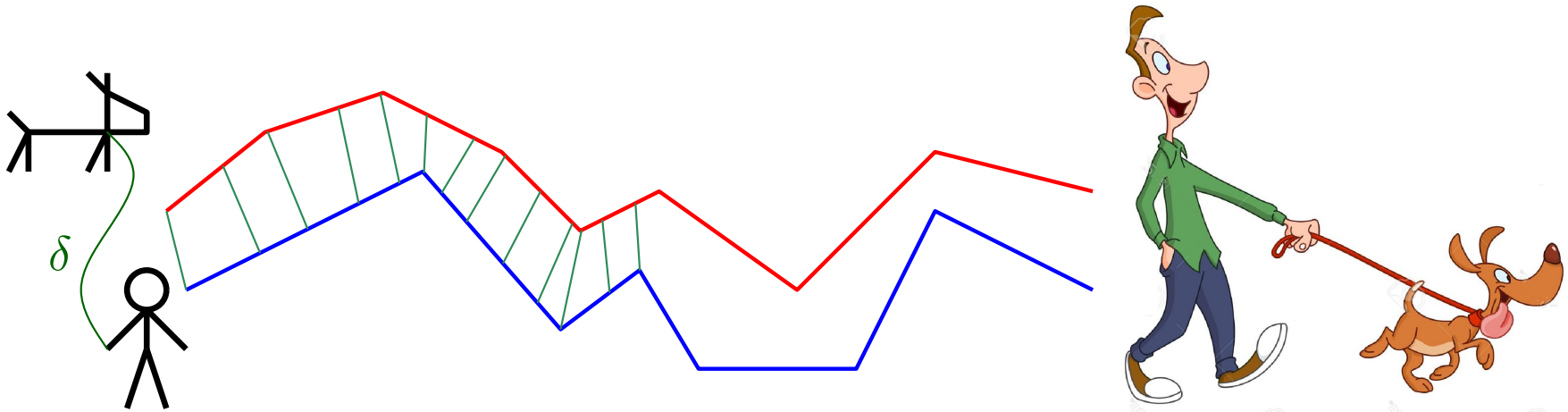

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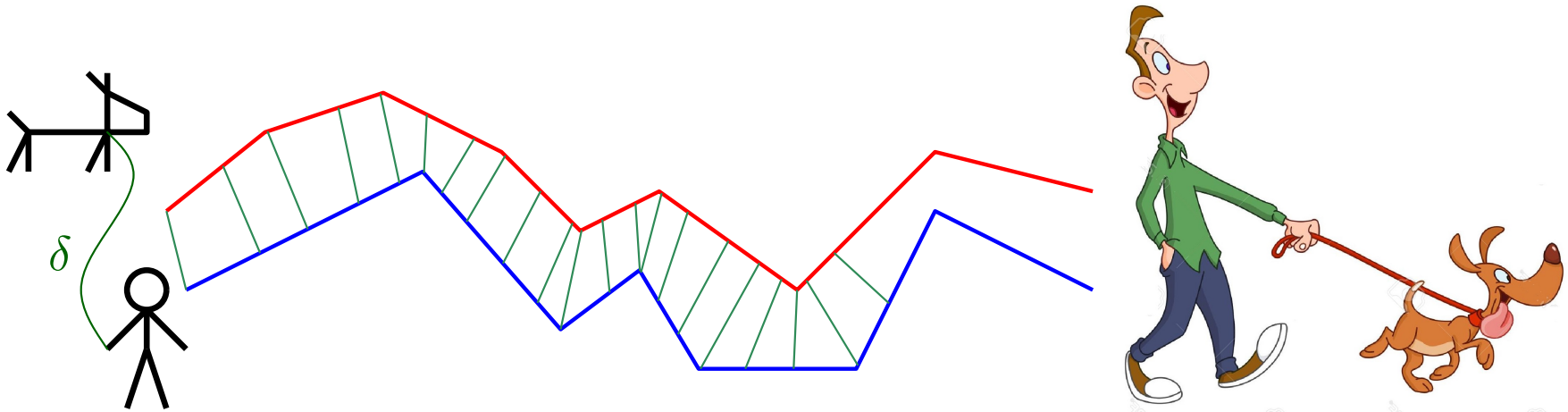

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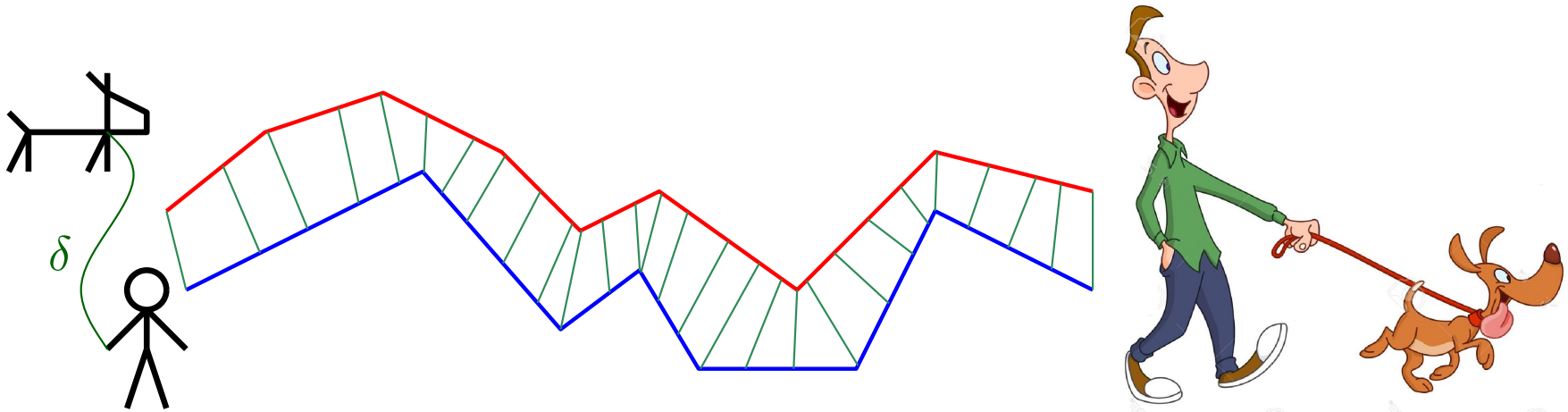

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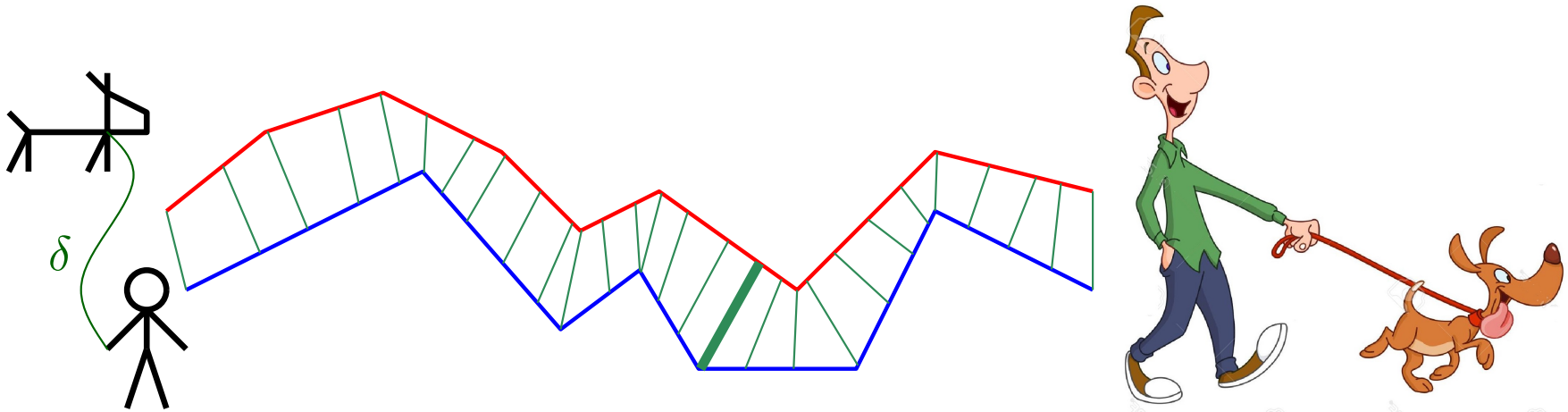
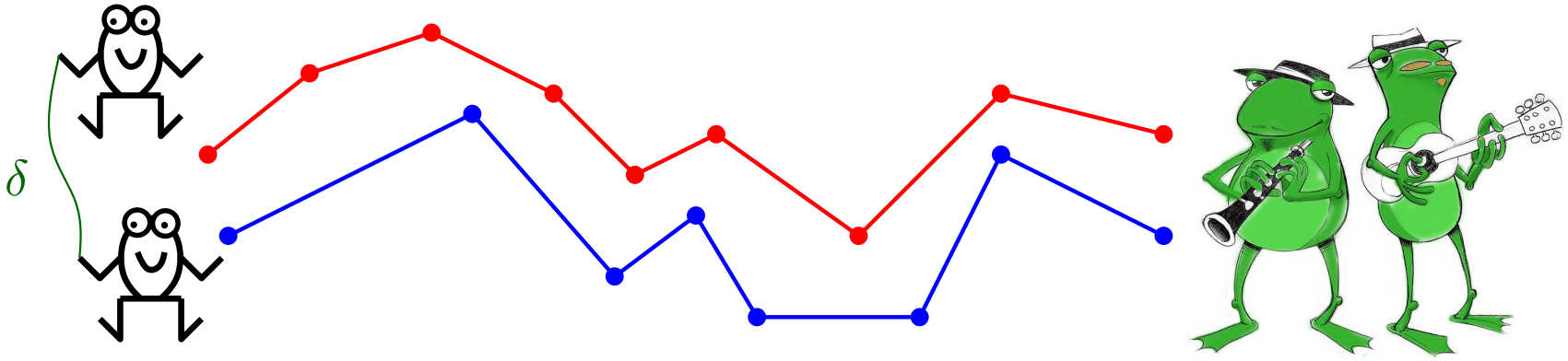
- The **Fréchet distance** ($d_F(A, B)$) is the **minimum** δ that is sufficient for traversing both curves in this manner.



The discrete Fréchet distance

Two sequences of points, $A = (a_1, ..., a_n)$ and $B = (b_1, ..., b_m)$.
Two frogs, the A-frog and the B-frog, connected by a leash of length δ, **hopping** along their respective sequences.



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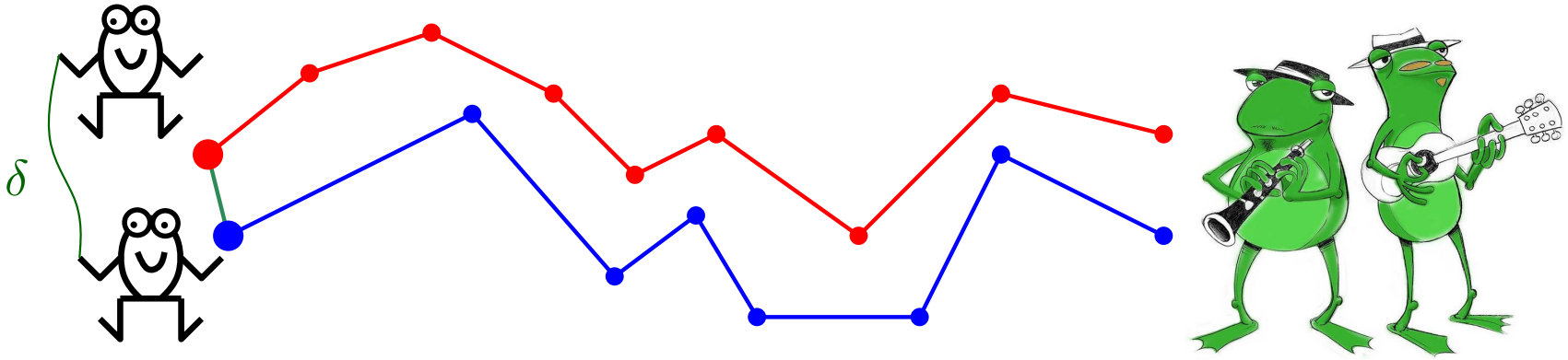

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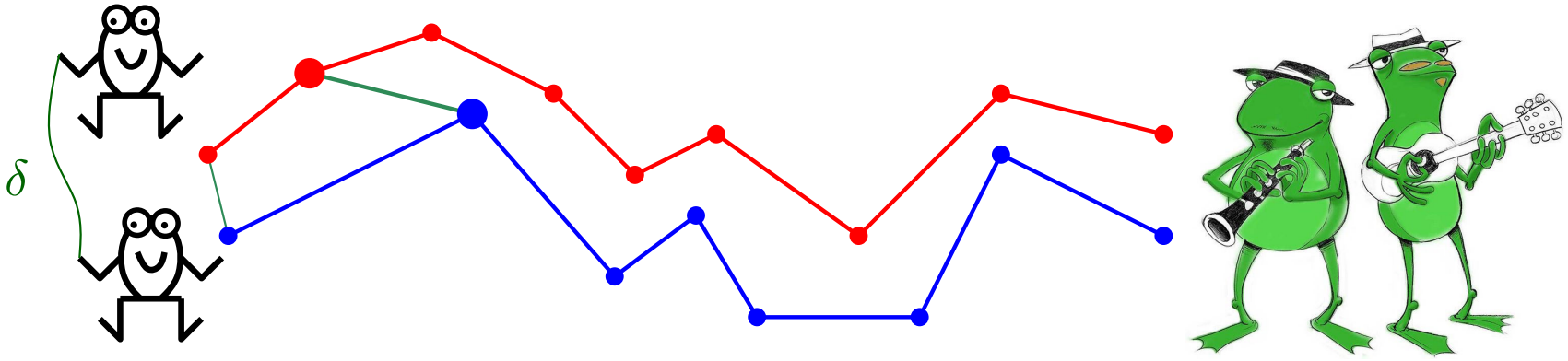

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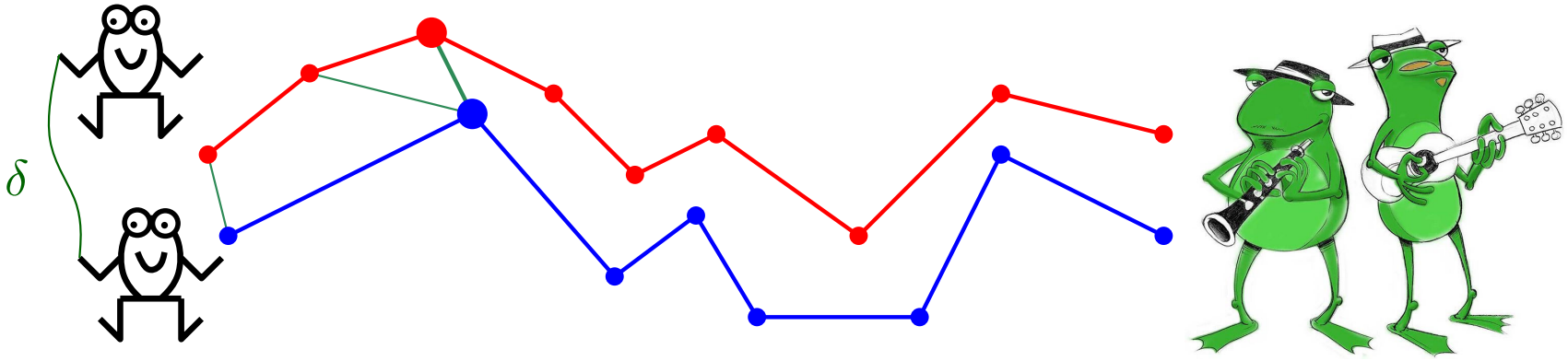

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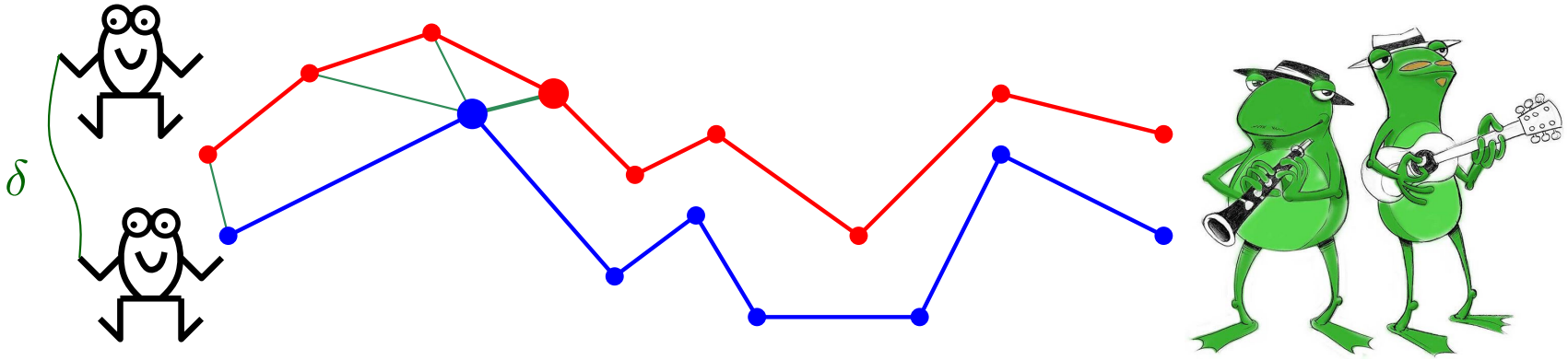

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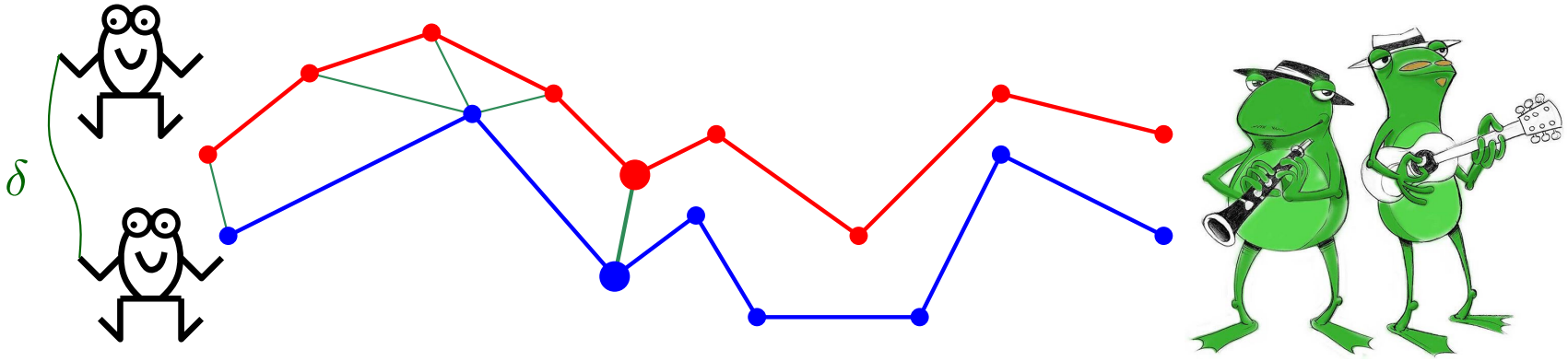

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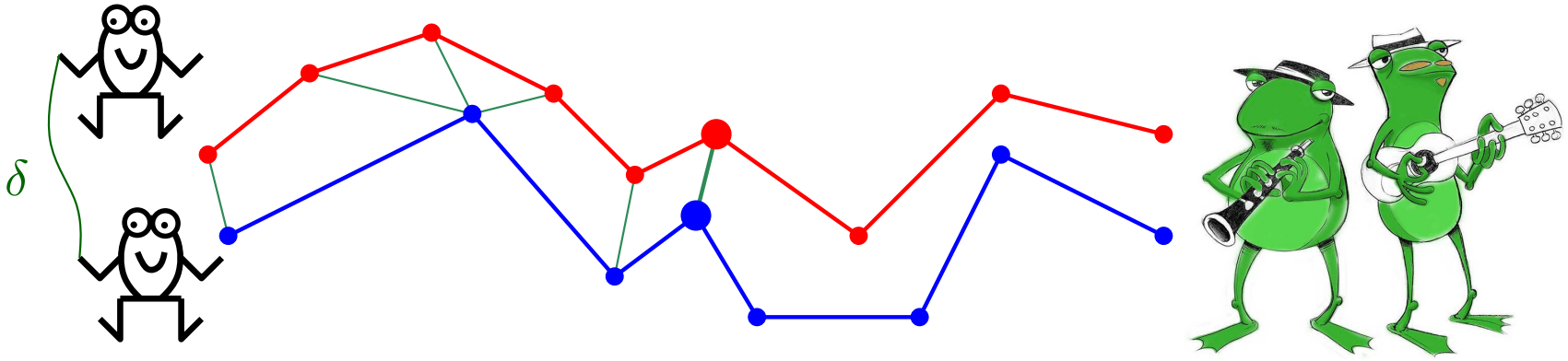

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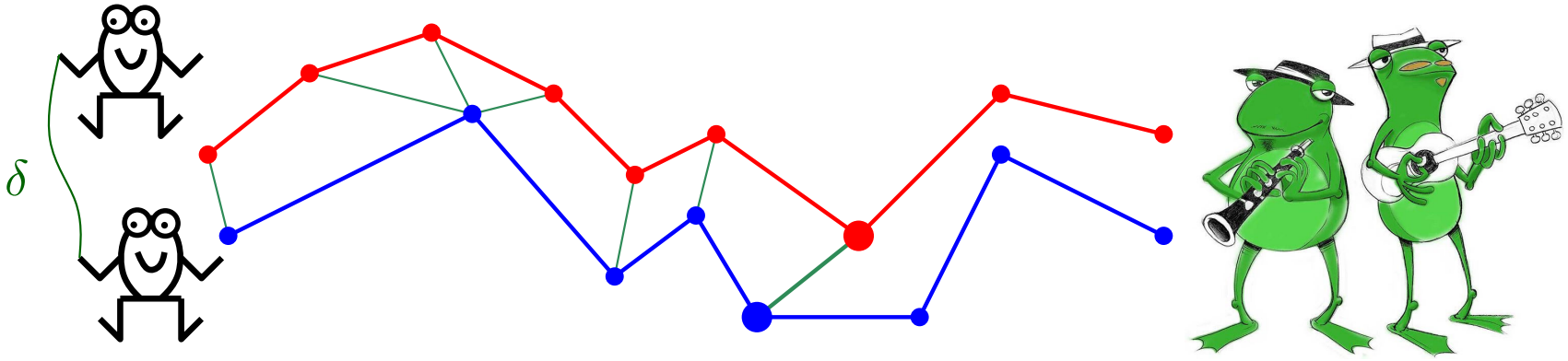

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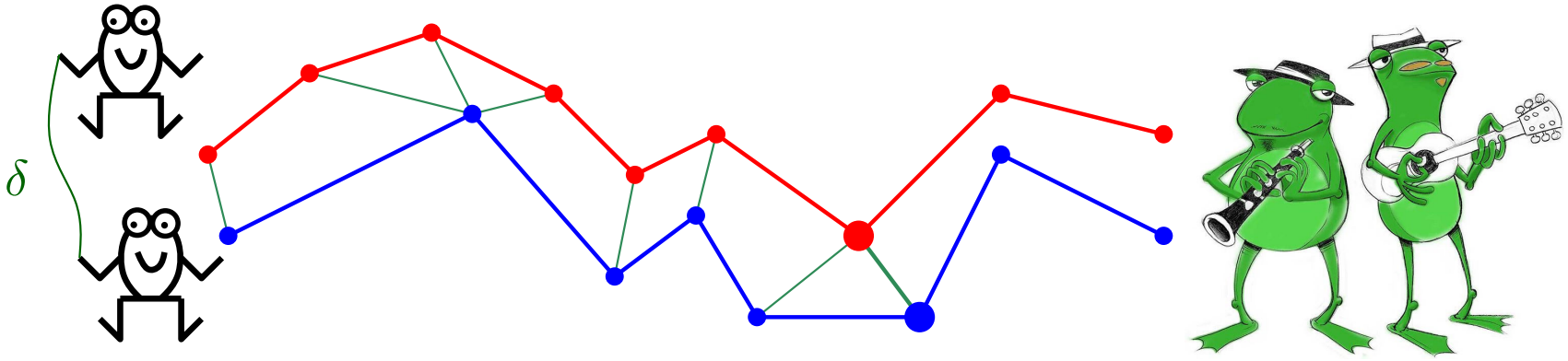

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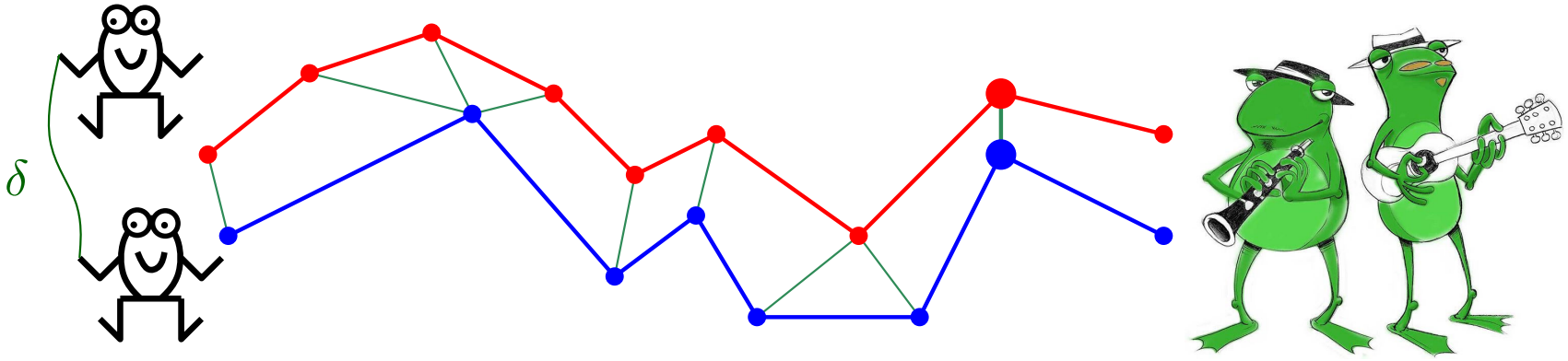

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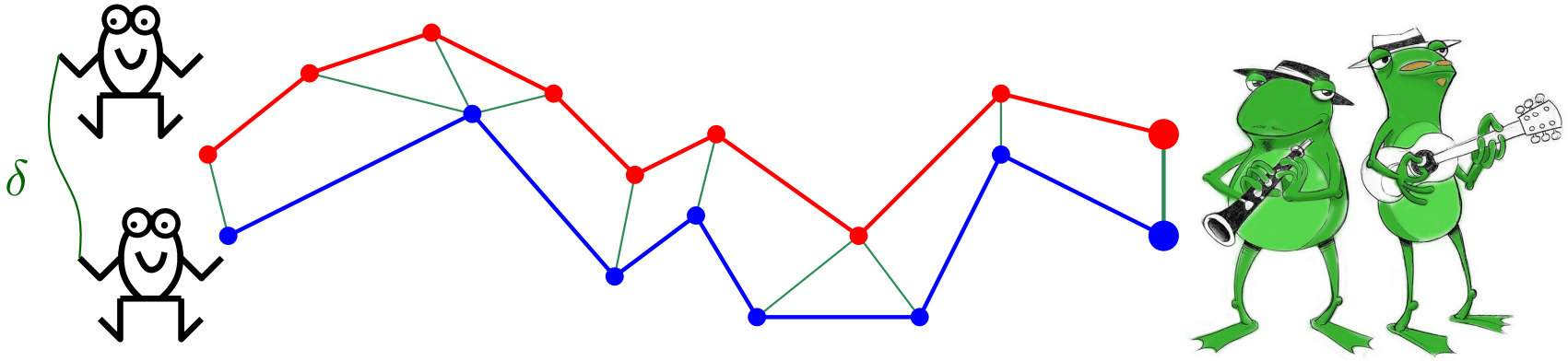

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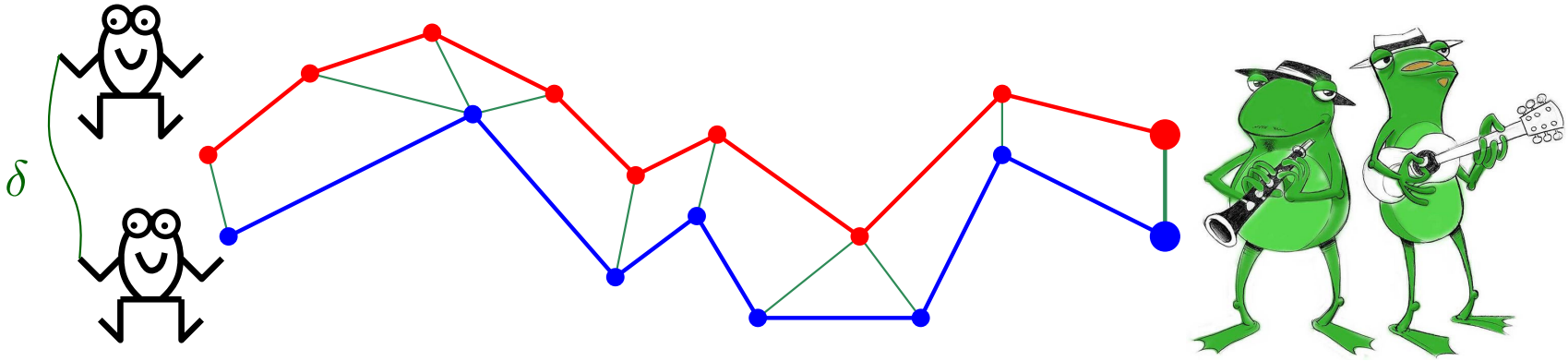
- The **discrete Fréchet distance** ($d_{dF}(A, B)$) is the **minimum** δ that allows the frogs to reach a_n and b_m.



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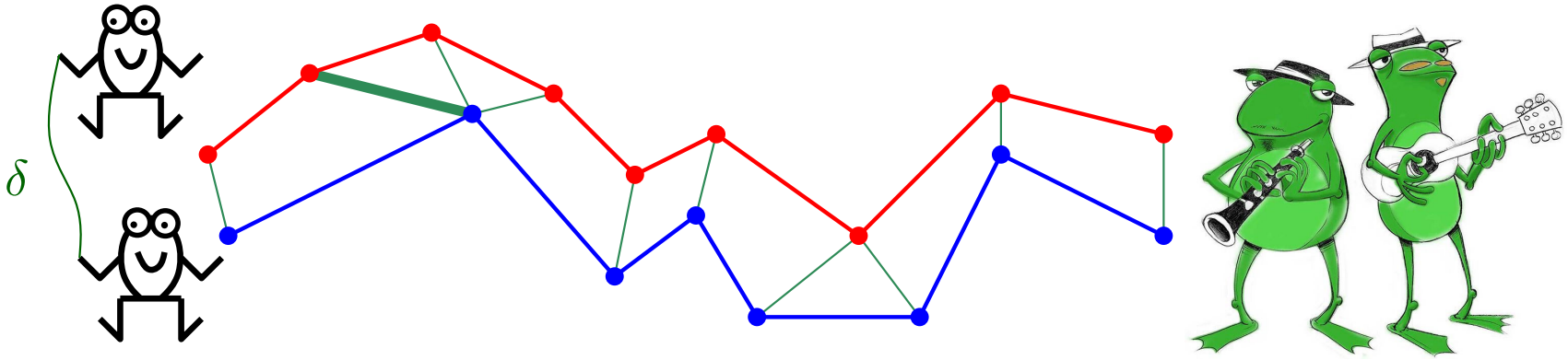

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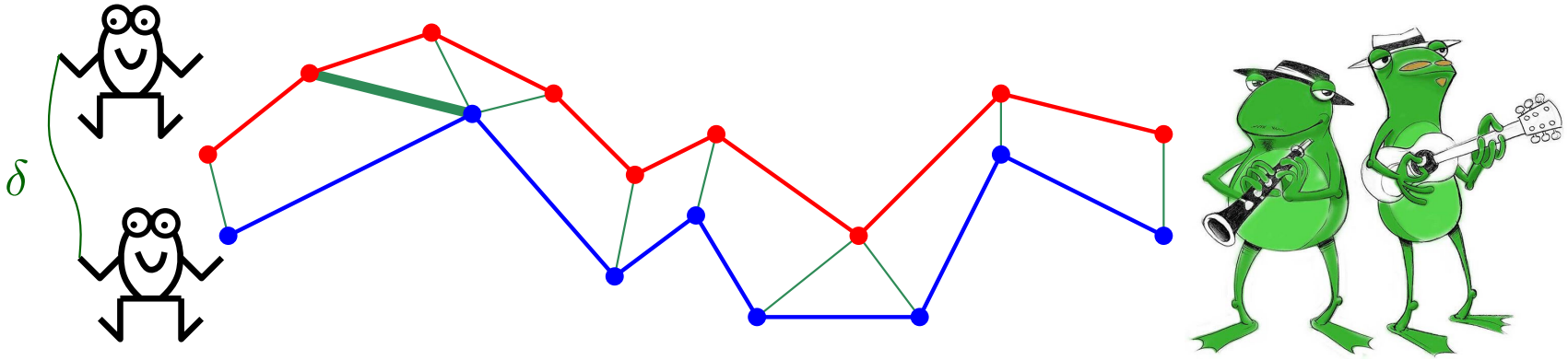
- The **discrete Fréchet distance** ($d_{dF}(A, B)$) is the **minimum** δ that allows the frogs to reach a_n and b_m.

- A good approximation of the continuous distance.
- Makes more sense in some situations (computational biology).

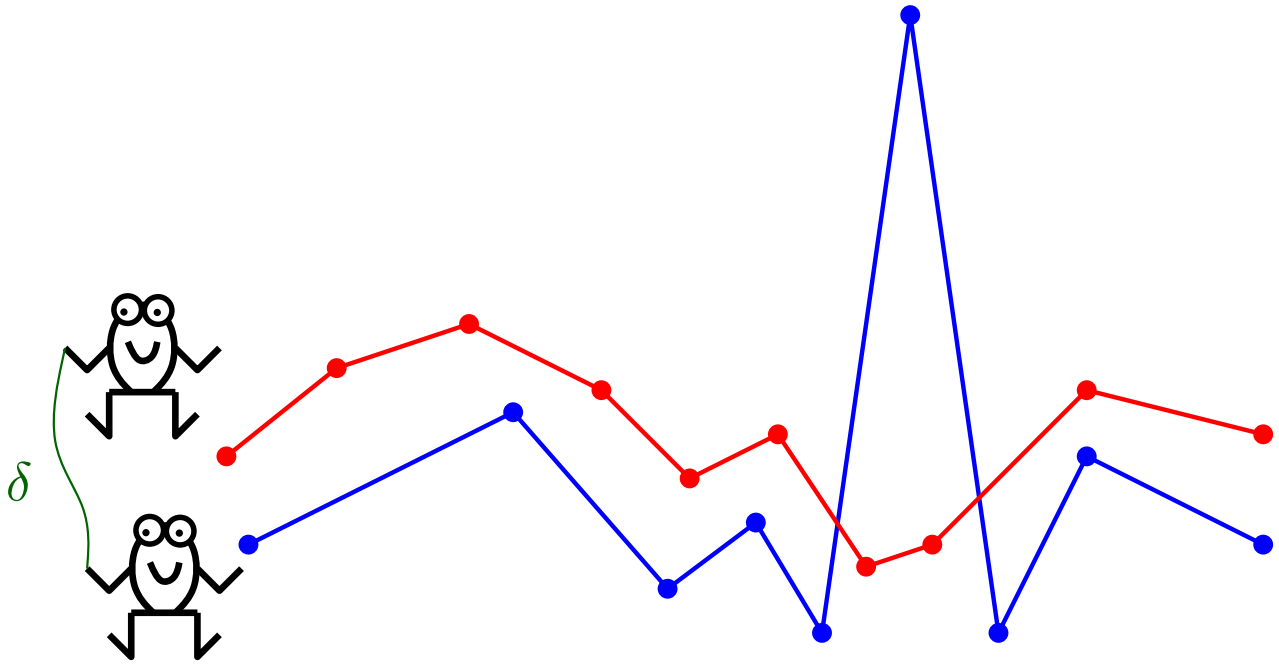


Related work

- **Eiter and Mannila ('94)** showed that $d_{dF}(A, B)$ can be computed in $O(n^2)$ time.
- **Agarwal et al. ('13)** showed how to compute $d_{dF}(A, B)$ in $O\left(\frac{n^2 \log \log n}{\log n}\right)$ time.
- **Bringmann and Mulzer ('15)** presented a conditional lower bound that **strongly subquadratic algorithms** for the discrete Fréchet distance **are unlikely to exist**, even in the one-dimensional case and even if the solution may be approximated up to a factor of 1.399.

Handling outliers

Physical sensors, such as GPS, may generate inaccurate measurements, which we refer to as **outliers**.



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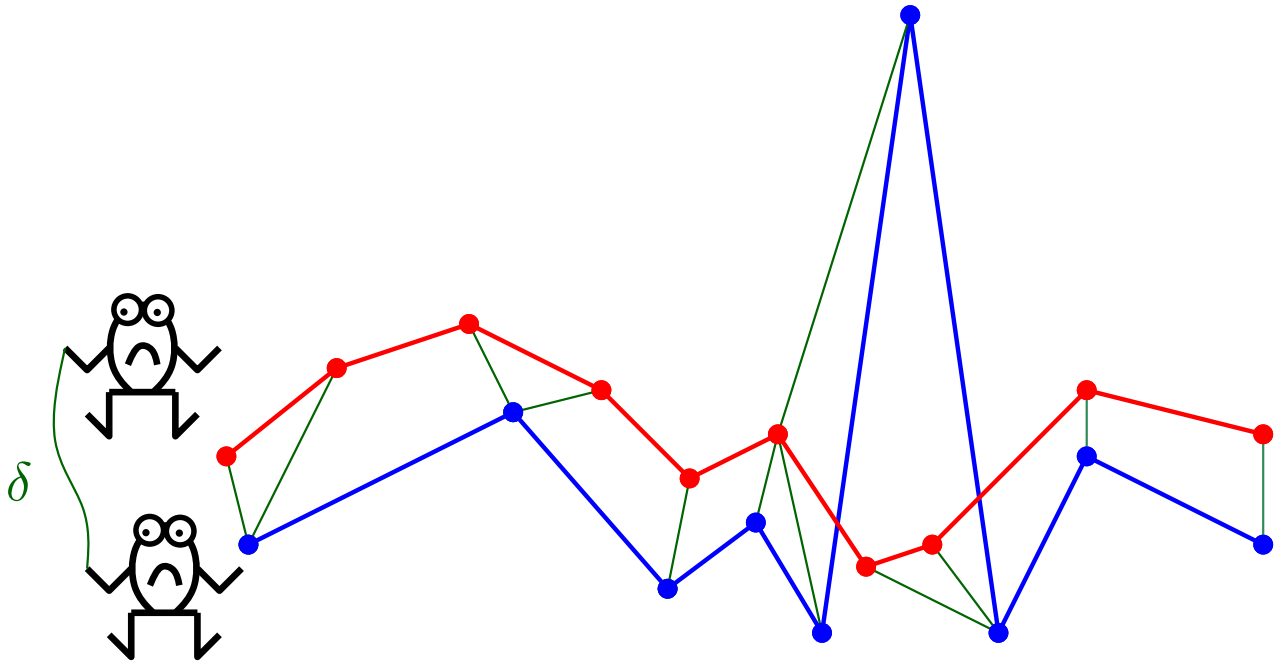
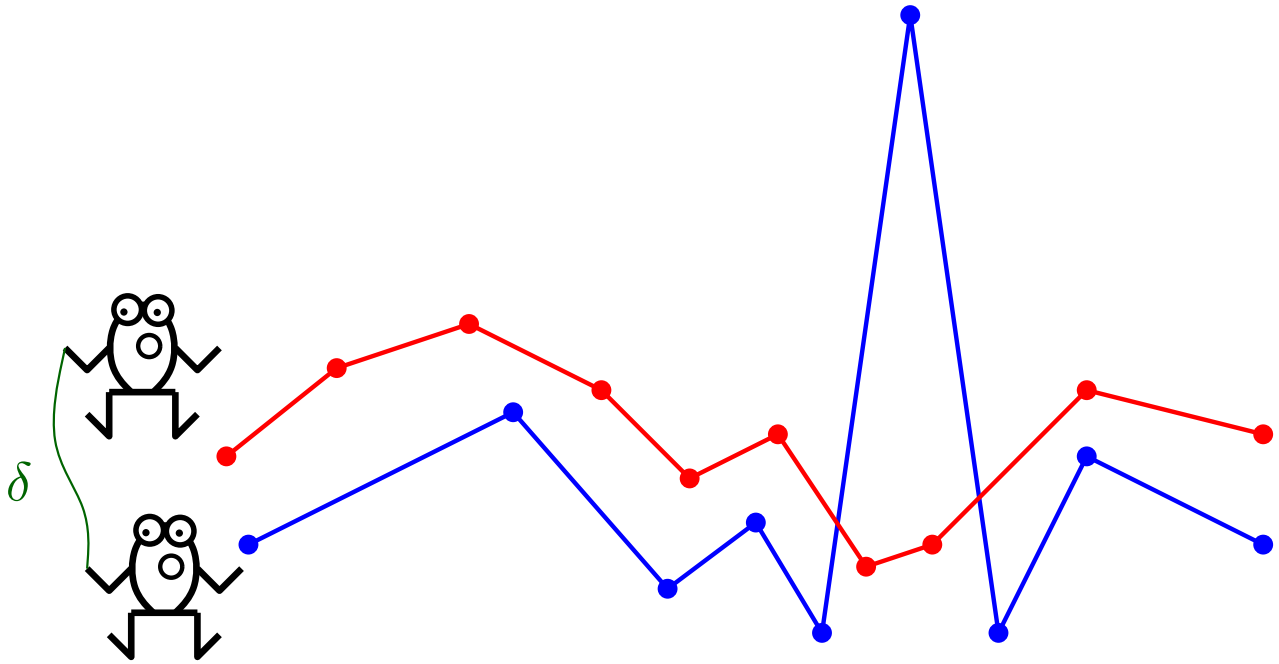
- The Fréchet distance and the discrete Fréchet distance are sensitive to outliers.



Handling outliers

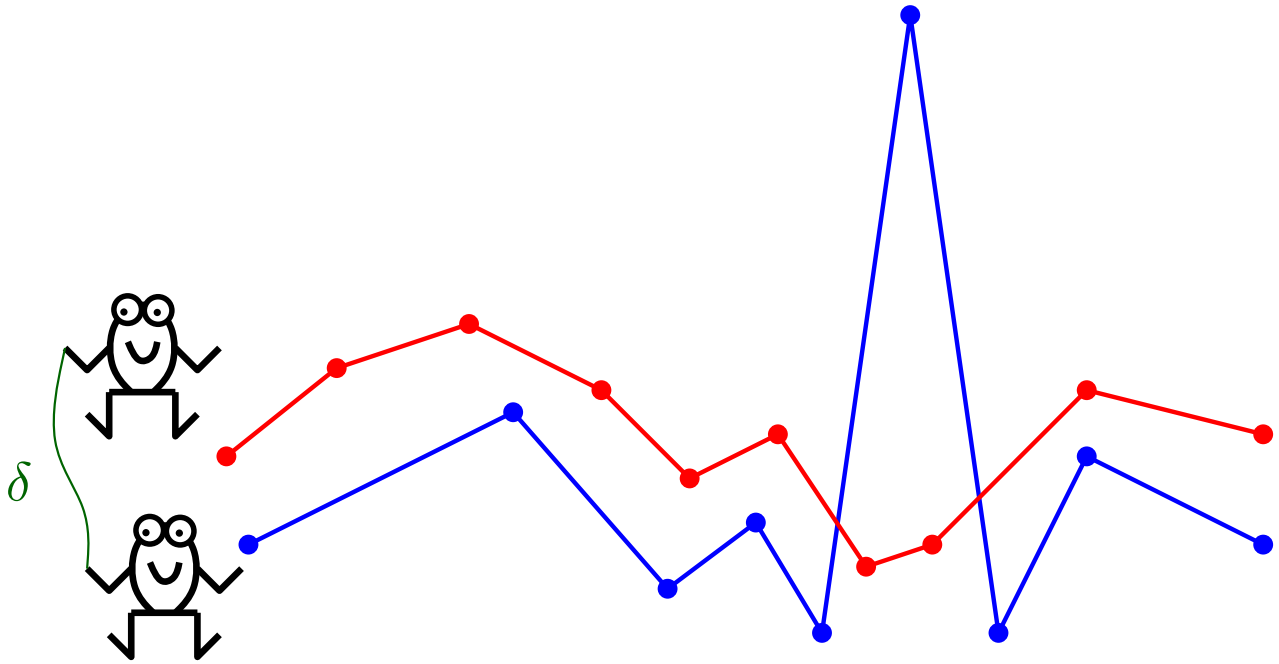
How to reduce sensitivity to outliers?



Handling outliers

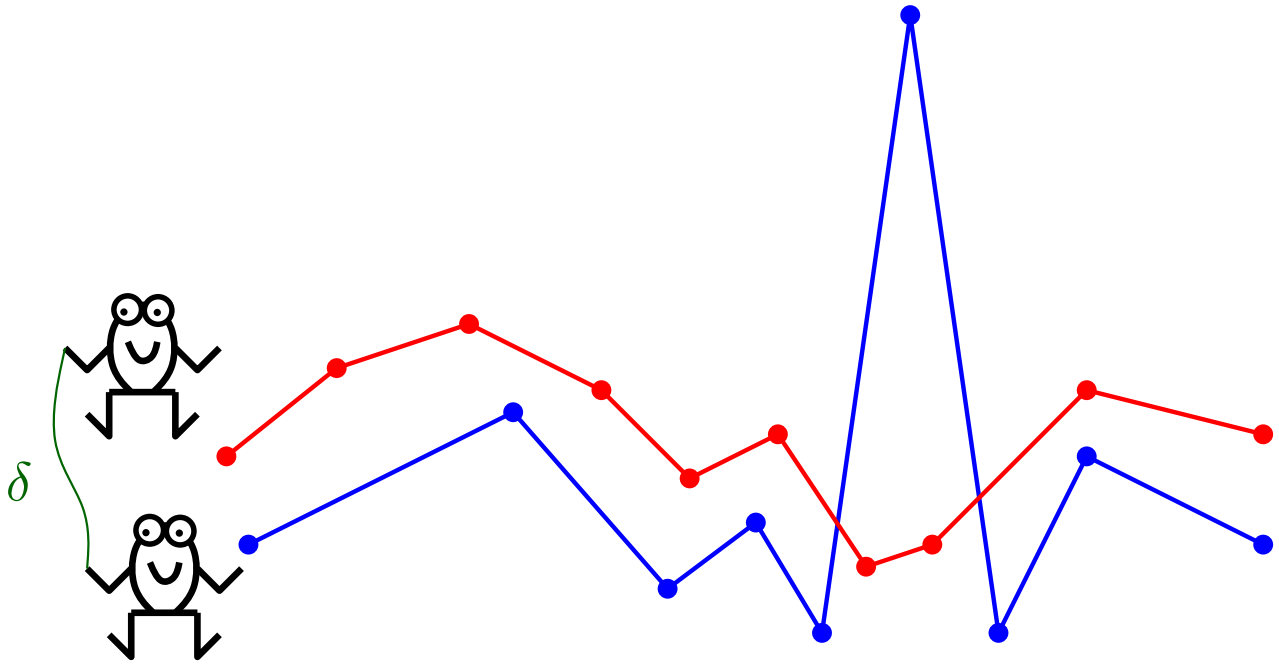
How to reduce sensitivity to outliers?
Take **shortcuts**!

- Allow the A-frog or the B-frog (or both) to "ignore" subcurves or points which might be considered as noise.



The one-sided discrete Fréchet distance with shortcuts

$A = (a_1, ..., a_n)$, $B = (b_1, ..., b_m)$, a leash of length δ.
Only the A-frog can skip points.



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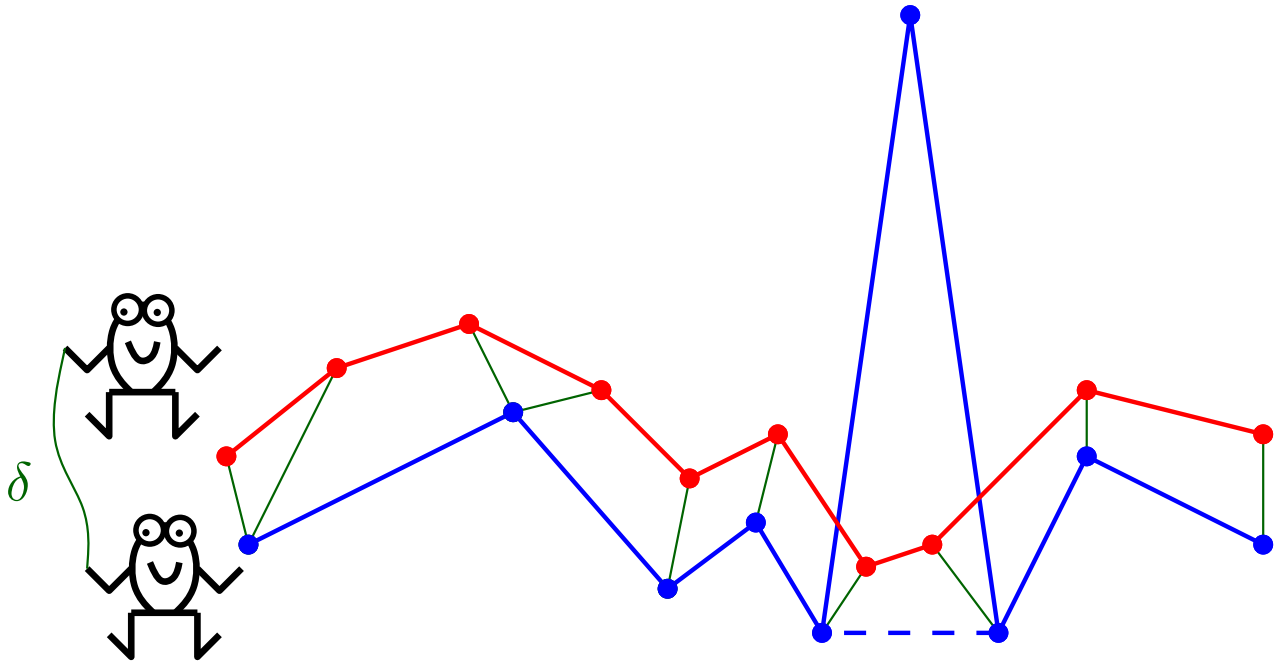
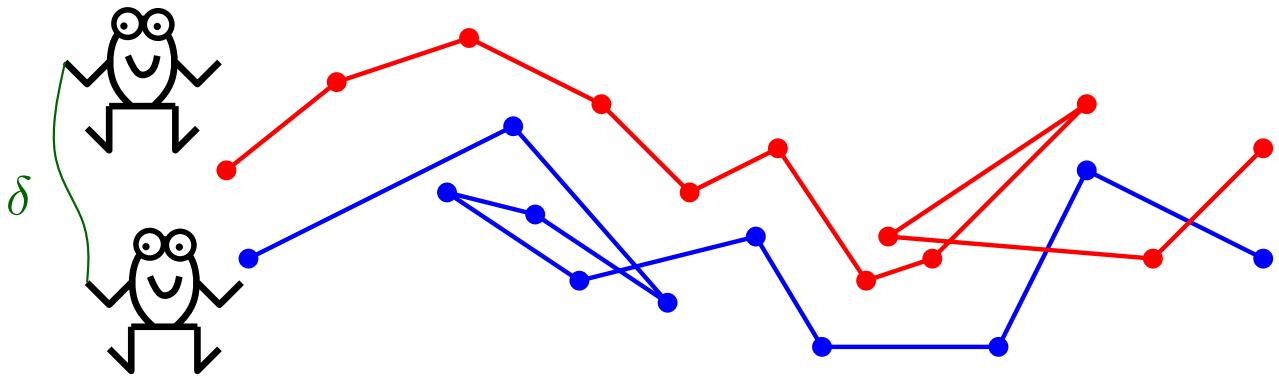
- The **one-sided discrete Fréchet distance with shortcuts** is the **minimum** such δ that allows the frogs to reach a_m and b_n.



The weak discrete Fréchet distance

$A = (a_1, ..., a_n)$, $B = (b_1, ..., b_m)$, a leash of length δ.
The frogs are also allowed to jump (one step) backwards.



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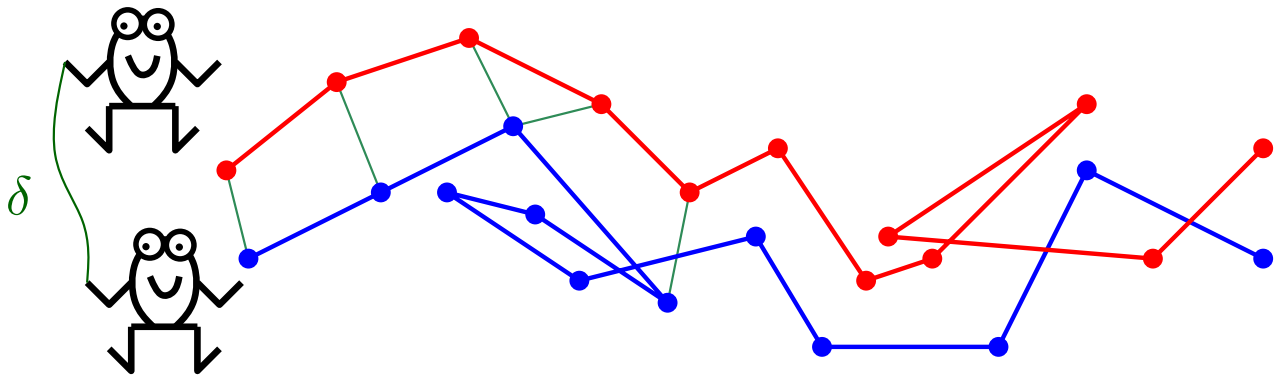

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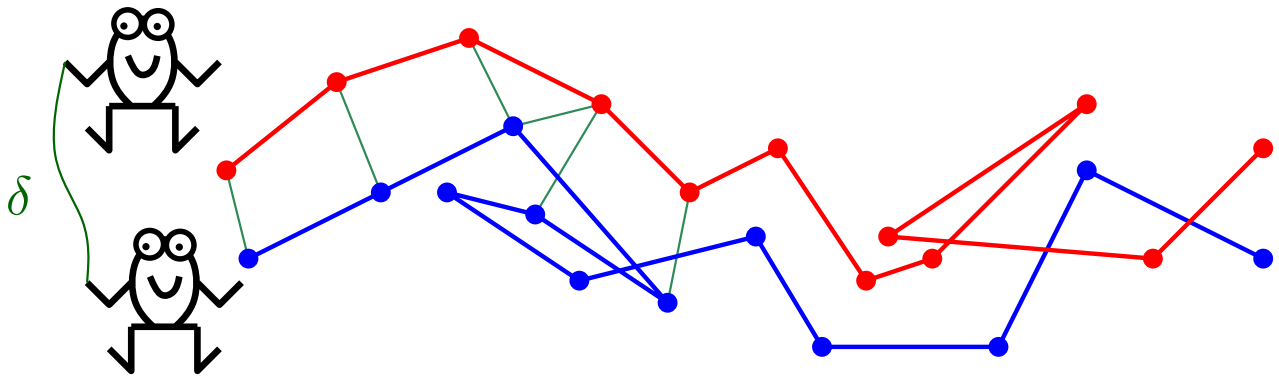

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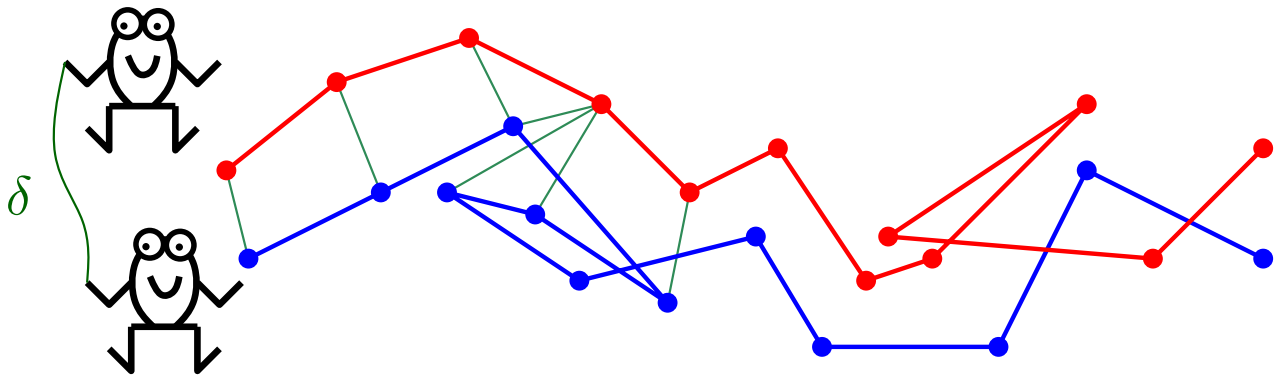

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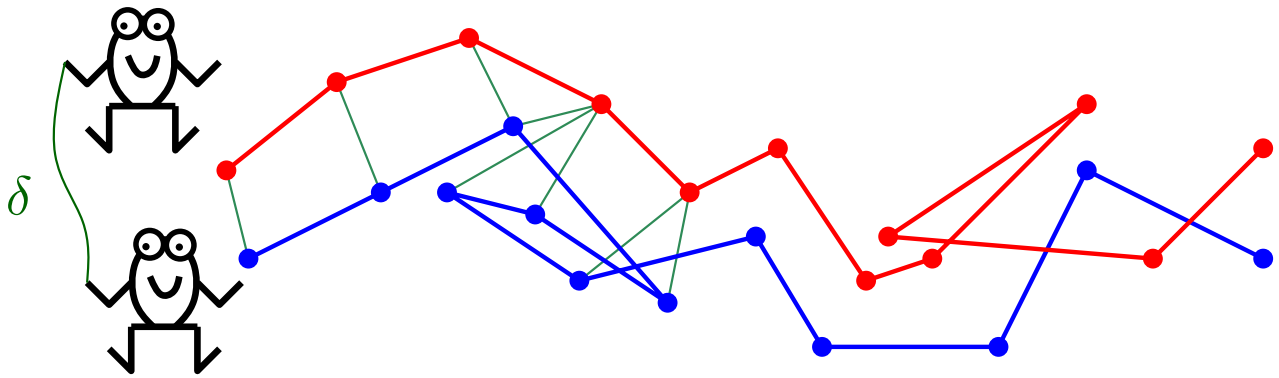

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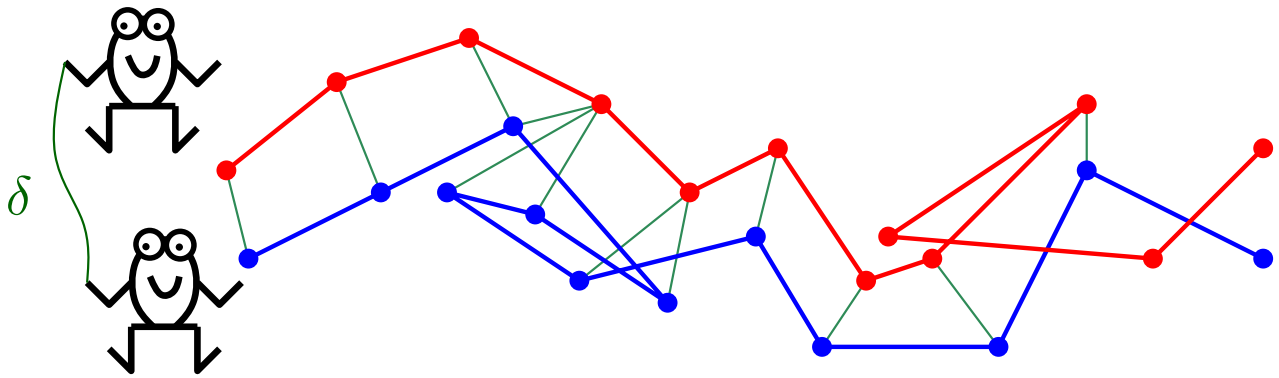

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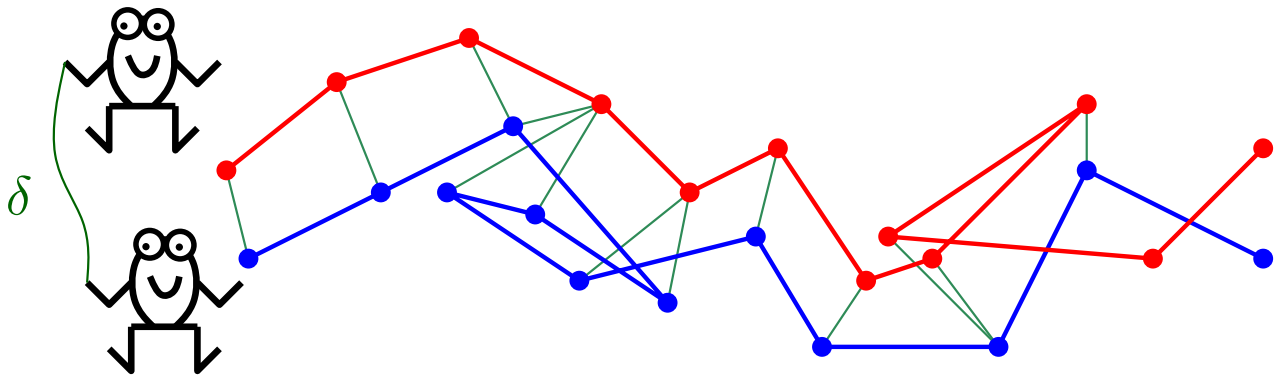
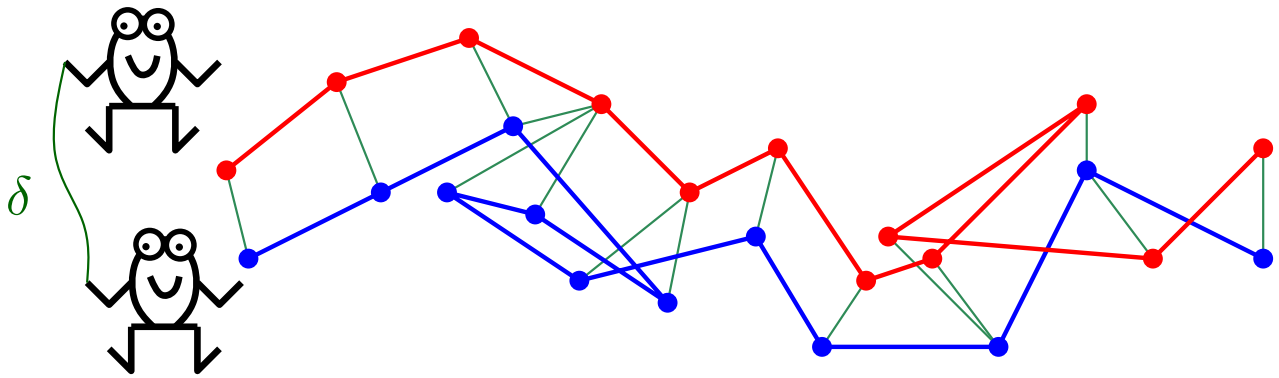


The weak discrete Fréchet distance

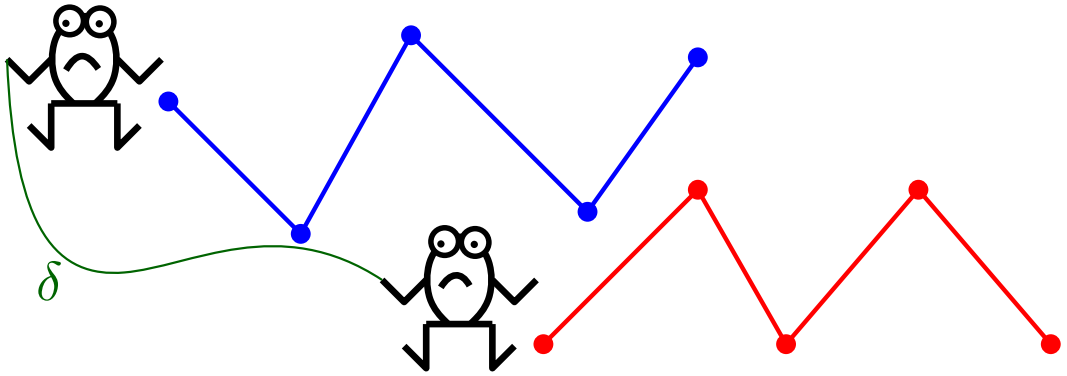
$A = (a_1, ..., a_n)$, $B = (b_1, ..., b_m)$, a leash of length δ.
The frogs are also allowed to jump (one step) backwards.

- The **weak discrete Fréchet distance** is the **minimum** such δ that allows the frogs to reach a_m and b_n.



Fréchet under translation

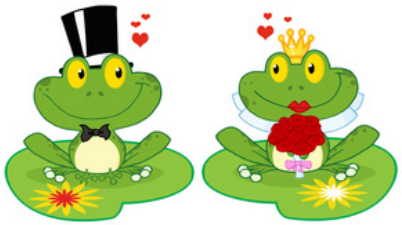
The input curves are not necessarily **aligned**, and one of them needs to be **adjusted**.

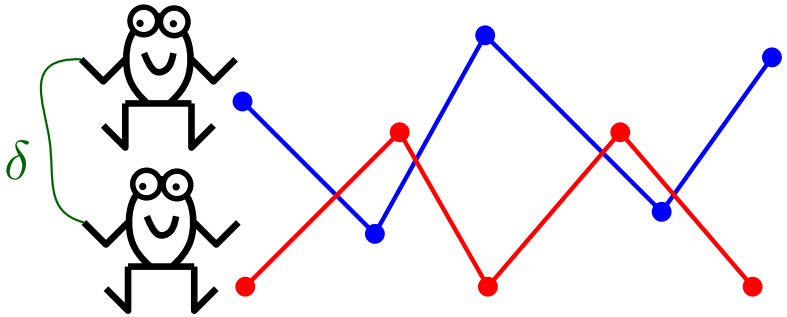


Fréchet under translation

The input curves are not necessarily **aligned**,
and one of them needs to be **adjusted**.

- Given $A = (a_1, ..., a_n)$, $B = (b_1, ..., b_m)$, find a translation t that minimizes the discrete Fréchet distance between A and $B + t$.





Related work

Continuous Fréchet distance under translation:

- **Alt et al. ('01)**: $O(m^3 n^3 (m+n)^2 \log(m+n))$-time algorithm for points in 2D, and an algorithm computing a $(1+\epsilon)$-approximation in $O(\epsilon^{-2} mn)$ time.
- **Wenk ('03)**: $O((m+n)^{11} \log(m+n))$-time algorithm for points in 3D (general results for d dimensions and other families of transformations).

Related work

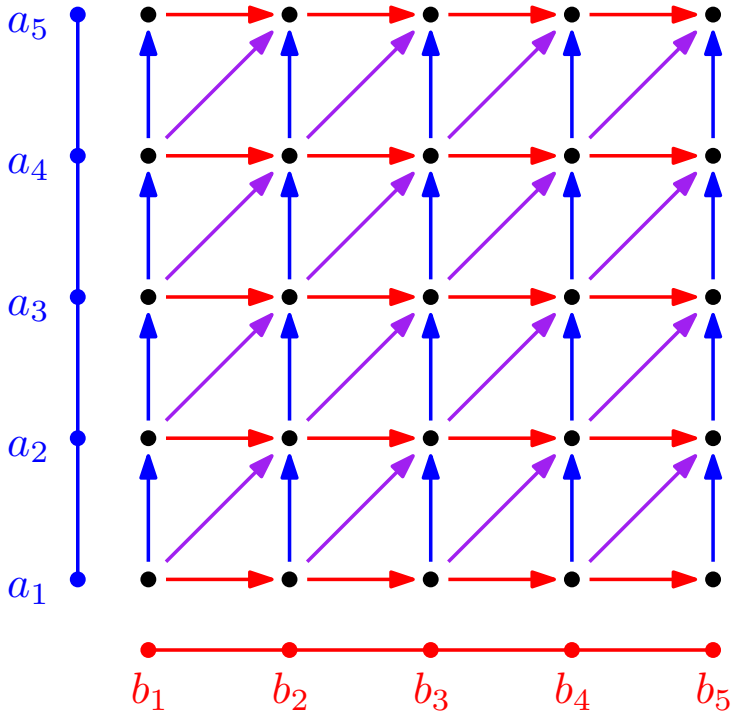
Discrete Fréchet distance under translation:

- **Mosig et al. ('05)**: $O(m^2n^2)$-time approximation algorithm for DFD under translation, rotation and scaling in 2D, with approximation factor close to 2.
- **Jiang et al. ('08)**: $O(m^3n^3 \log(m+n))$-time algorithm for DFD under translation, and an $O(m^4n^4 \log(m+n))$-time algorithm when both rotations and translations are allowed.
- **Ben Avraham et al. ('15)**: $O(m^3n^2(1+\log(\frac{n}{m}))\log(m+n))$-time algorithm, based on a dynamic data structure which supports updates and reachability queries in $O(m(1+\log(n/m))$ time (up next).

The positions graph

$G = G(V = A \times B, E = E_A \cup E_B \cup E_{AB})$

- V: all possible **positions** of the frogs.
- E: all possible **moves** between positions.



$E_A = \{\langle (a_i, b_j), (a_{i+1}, b_j) \rangle\}$

$E_B = \{\langle (a_i, b_j), (a_i, b_{j+1}) \rangle\}$

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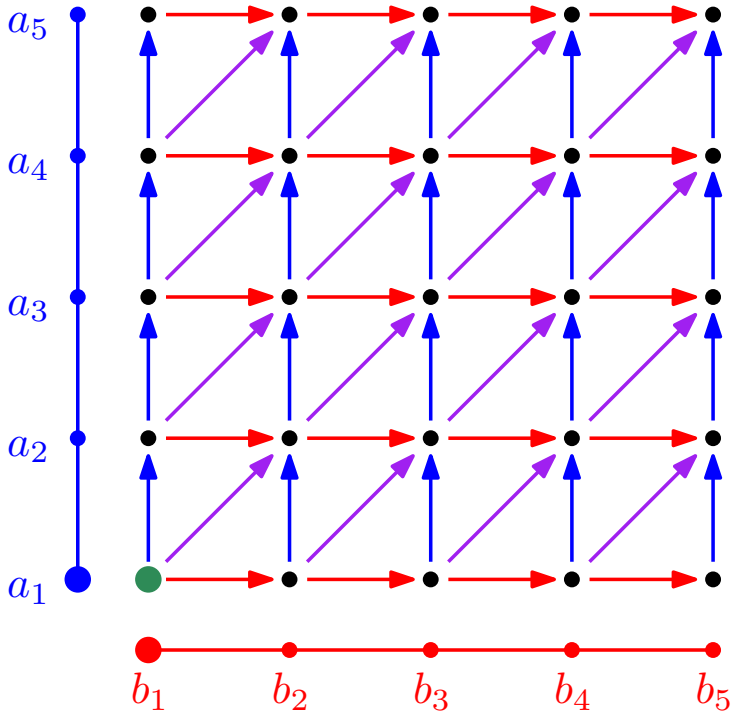

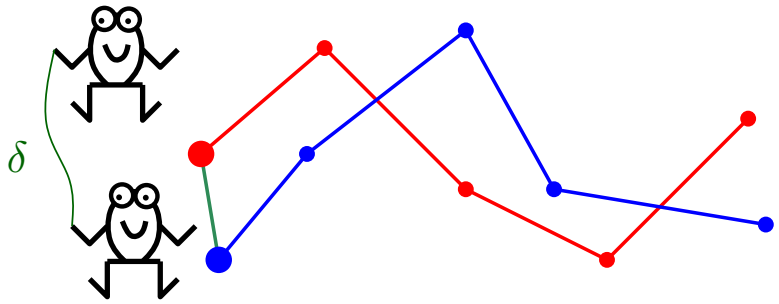

Sequence of moves (with unlimited leash length) $\Longleftrightarrow$
A path in G from (a_1, b_1) to (a_n, b_m).

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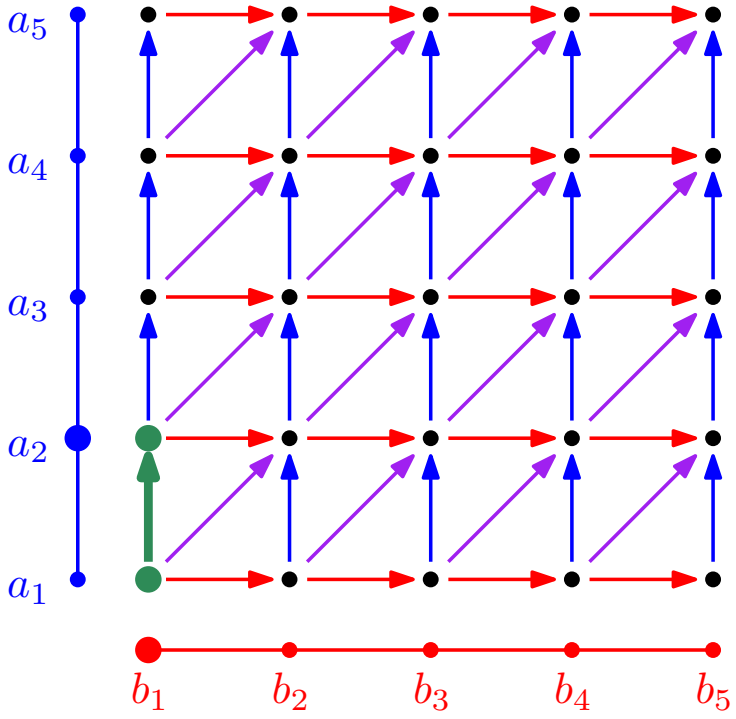

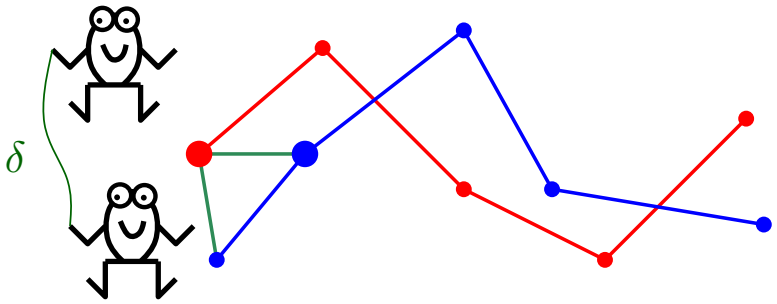

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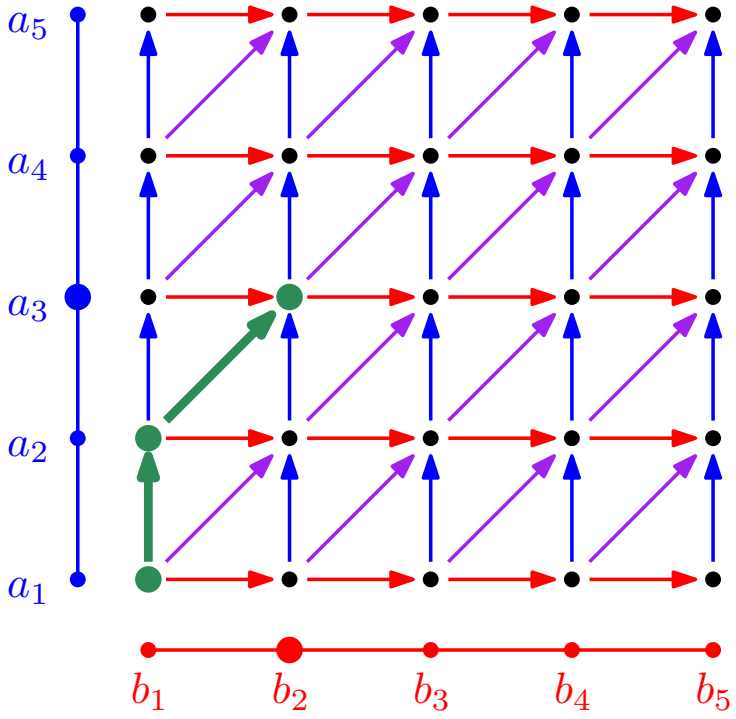

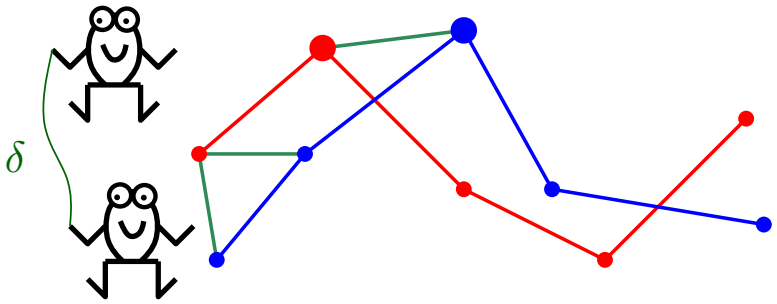

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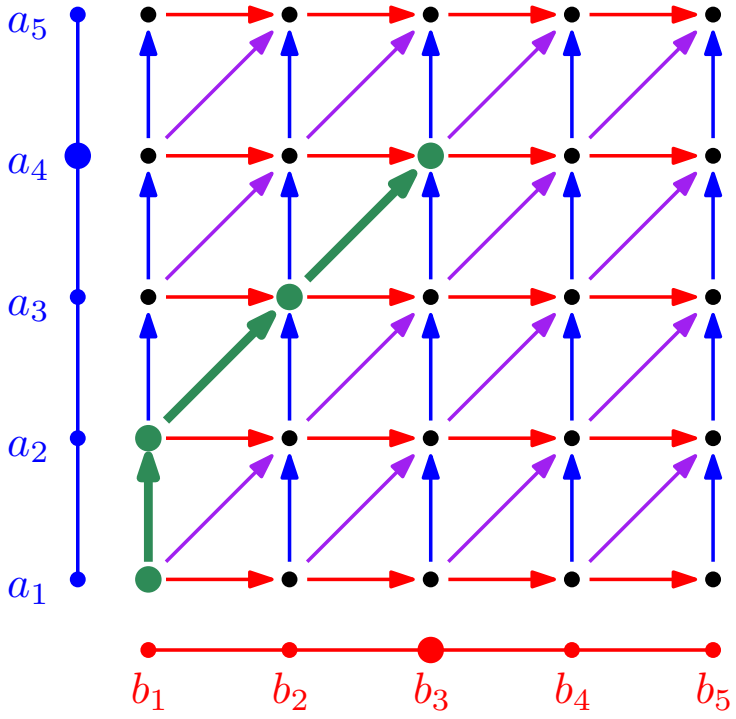

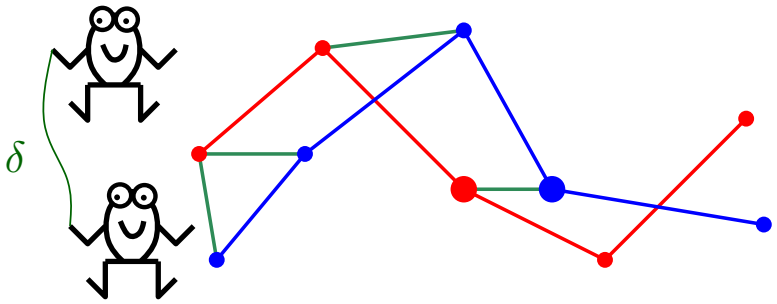

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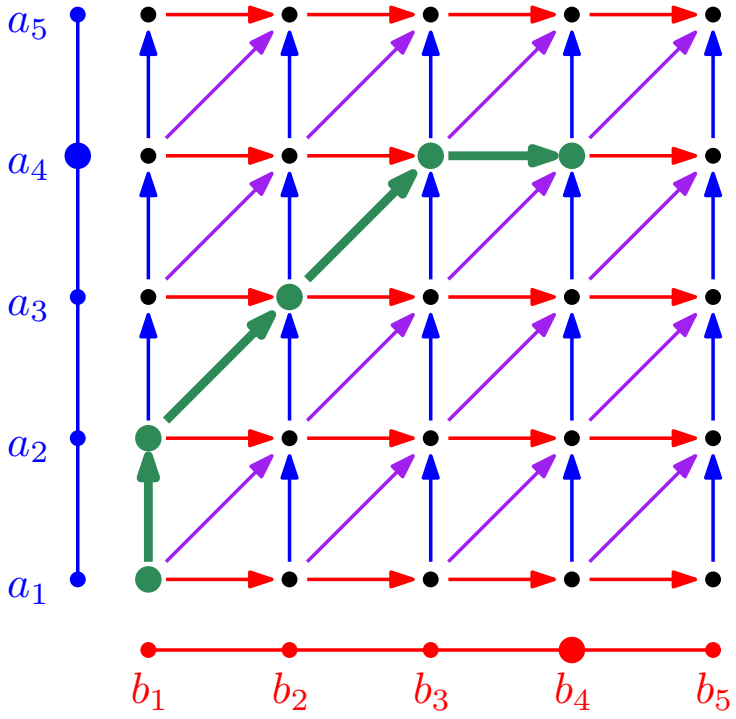

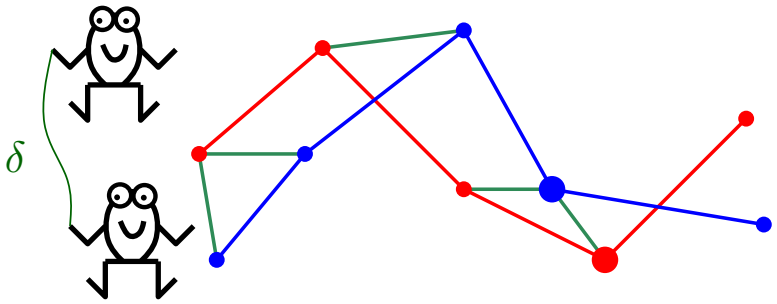

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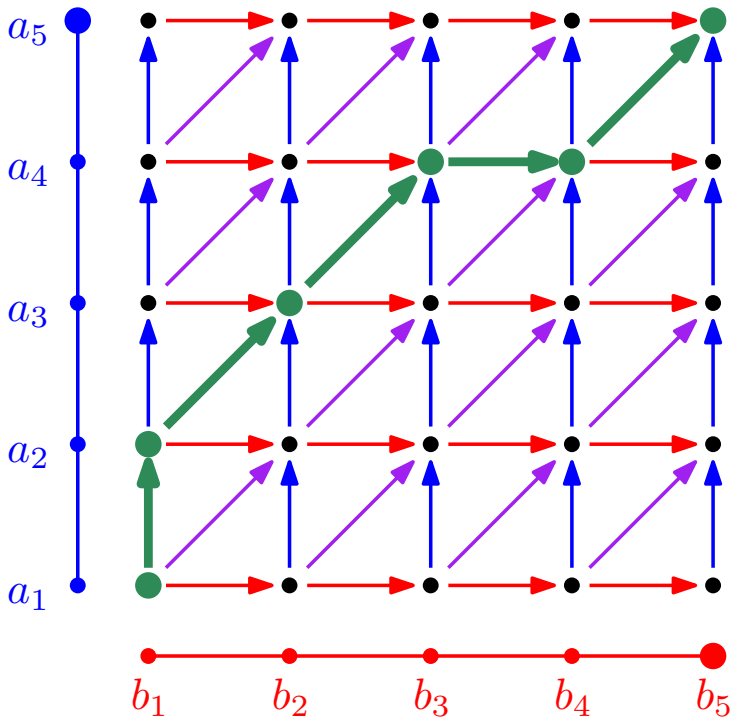

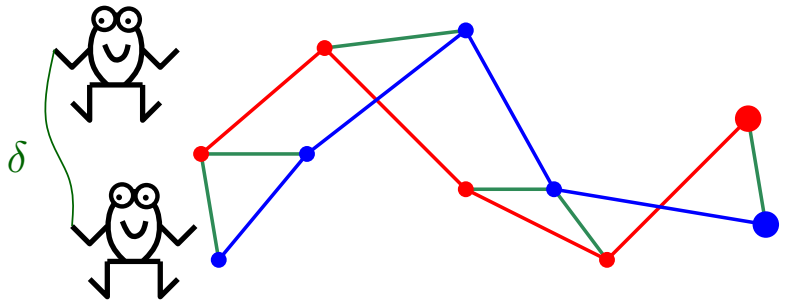

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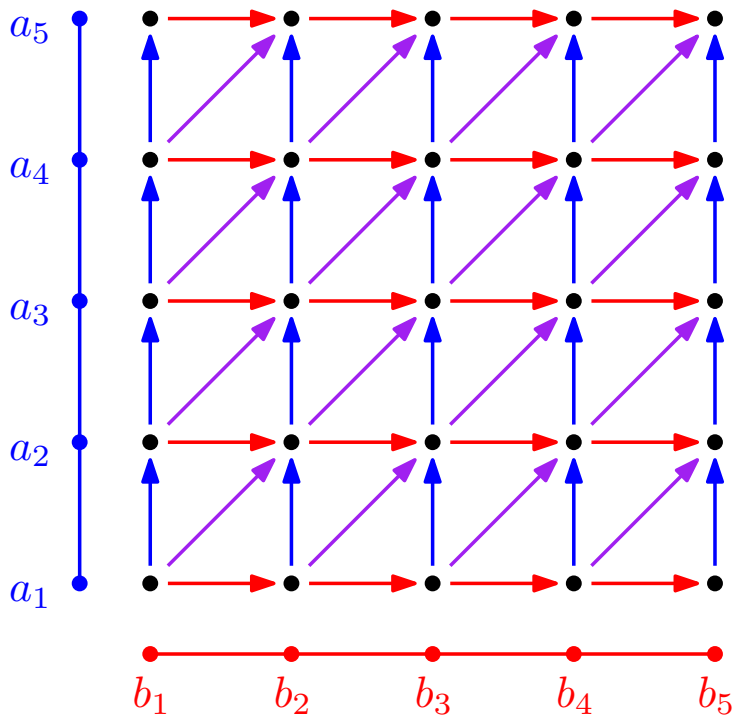
When the leash is short — not all positions in $A \times B$ are **valid**.



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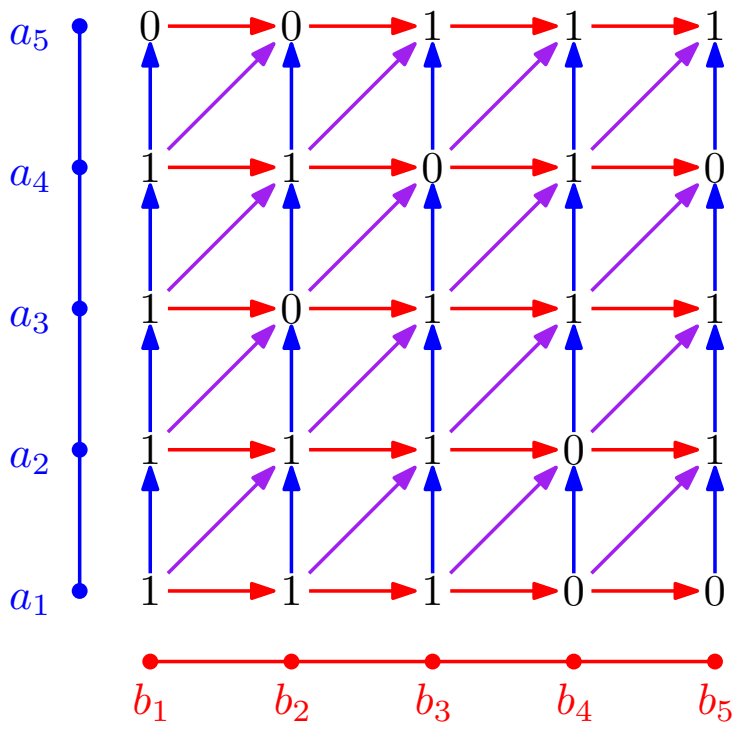
- **indicator** function $\sigma : A \times B \to \{0, 1\}$.



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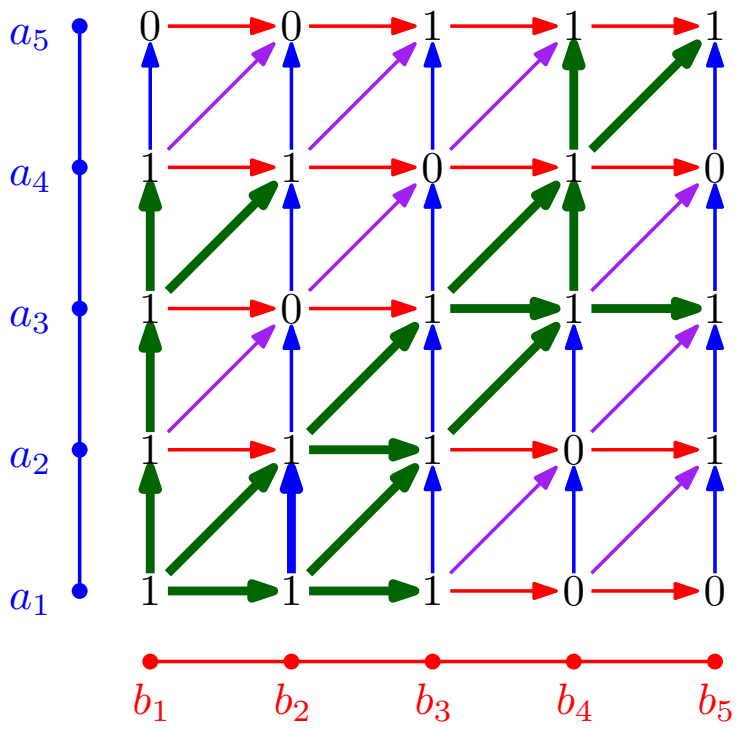
(a_i, b_j) is a **reachable position** (w.r.t. σ), if there exists a path P in G from (a_1, b_1) to (a_i, b_j), consisting of only valid positions.



Discrete Fréchet distance

Let $d(a_i, b_j)$ denote the **Euclidean distance** between a_i and b_j.

Our indicator function: given a distance $\delta \geq 0$,

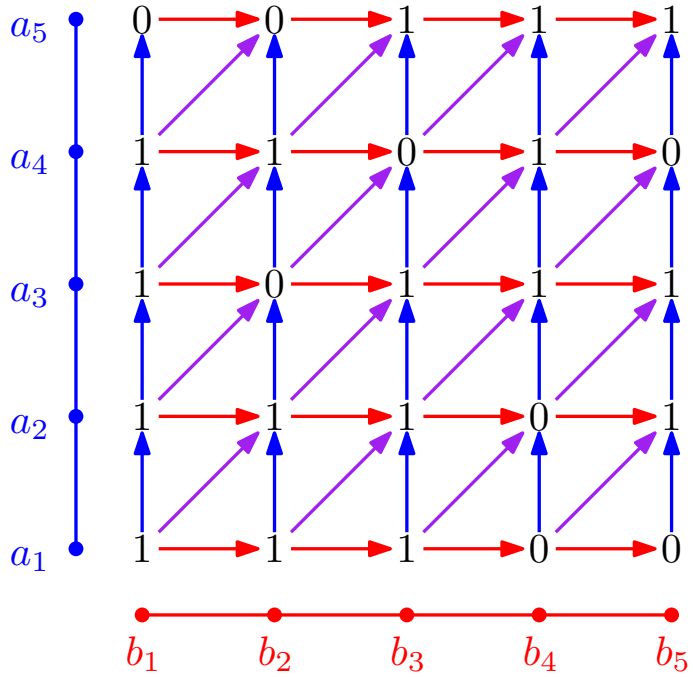
$$\sigma_\delta(a_i, b_j) = \begin{cases} 1, & d(a_i, b_j) \leq \delta \\ 0, & \text{otherwise} \end{cases}$$

The **discrete Fréchet distance** $d_{dF}(A, B)$ is the smallest $\delta \geq 0$ for which (a_n, b_m) is a reachable position w.r.t. σ_δ.

One-sided shortcuts

(a_i, b_j) is an **s-reachable position** (w.r.t. σ), if there exists a path P in G from (a_1, b_1) to (a_i, b_j), such that:

- $\sigma(a_1, b_1) = 1$ and $\sigma(a_i, b_j) = 1$.
- For each b_l, $1 < l < j$, there exists a position $(a_k, b_l) \in P$ that is valid (i.e., $\sigma(a_k, b_l) = 1$).



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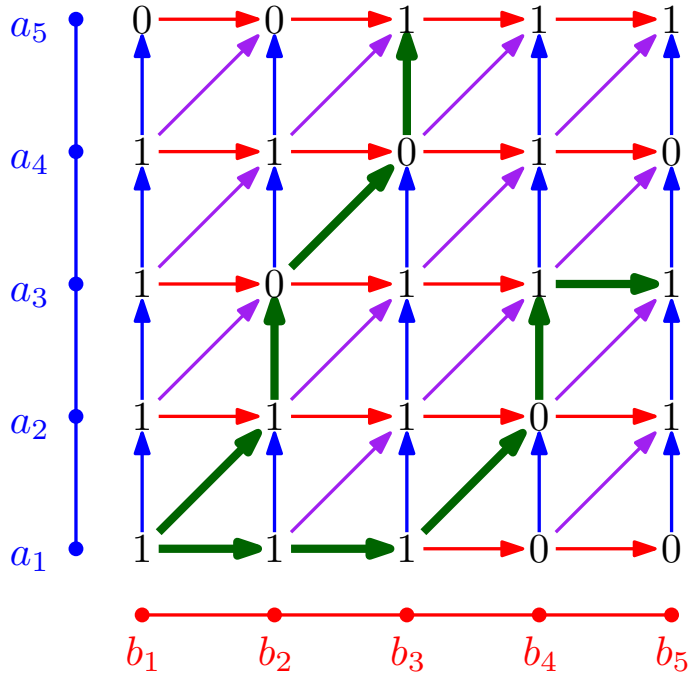

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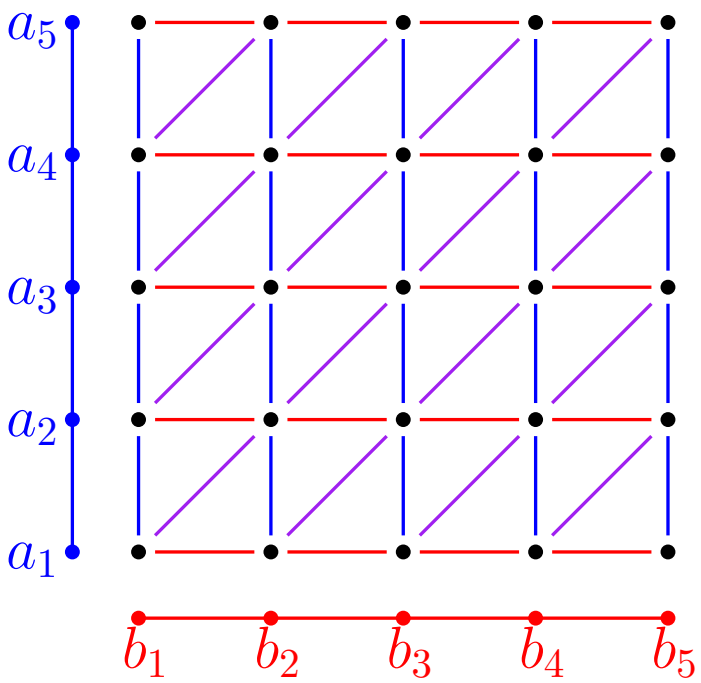
The **discrete Fréchet distance with shortcuts** $d^s_{dF}(A, B)$ is the smallest $\delta \geq 0$ for which (a_n, b_m) is an s-reachable position w.r.t. σ_δ.

Weak Fréchet distance

Backtracking is allowed!

Remove the directions from G, a new graph: $G_w = G(V = A \times B, E_w)$

- $E_w = \{(u, v) | \langle u, v \rangle \in E_A \cup E_B \cup E_{AB}\}$



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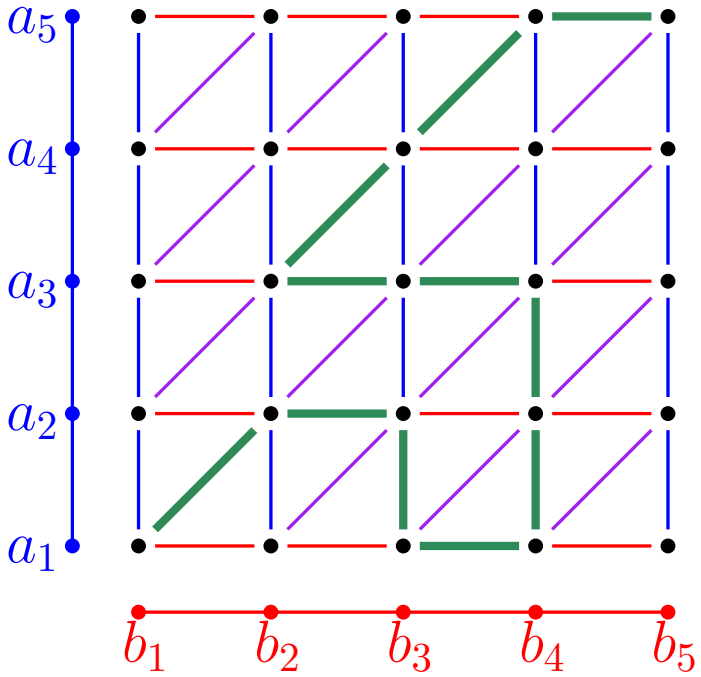

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The **weak discrete Fréchet distance** $d_{dF}^{w}(A, B)$ is the smallest $\delta \geq 0$ for which (a_n, b_m) is a w-reachable position w.r.t. σ_δ.

The translation problem

Given two sequences of points $A = (a_1, ..., a_n)$ and $B = (b_1, ..., b_m)$, find a translation t^* that minimizes $d_{dF}(A, B + t)$ over all translations t.

Denote:
$\widehat{d_{dF}}(A, B) = \min_t\{d_{dF}(A, B + t)\}$
$\widehat{d^s_{dF}}(A, B) = \min_t\{d^s_{dF}(A, B + t)\}$
$\widehat{d^w_{dF}}(A, B) = \min_t\{d^w_{dF}(A, B + t)\}$

The algorithm of Ben Avaraham, Kaplan, and Sharir

- **Ben Avraham et al. ('15)**: $O(m^3 n^2 (1 + \log(\frac{n}{m})) \log(m + n))$-time algorithm for the discrete Fréchet distance under translation, based on a dynamic data structure which supports updates and reachability queries in $O(m(1 + \log(n/m))$ time.

The algorithm of Ben Avaraham, Kaplan, and Sharir

STEP 1: Build a **dynamic data structure** for (discrete Fréchet) reachability queries.

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STEP 1: Build a **dynamic data structure** for (discrete Fréchet) reachability queries.

Given sequences A and B and an indicator function σ:

- The dynamic data structure is constructed in $O(mn)$ time.
- Allows the following operations in $O(m(1 + \log(n/m)))$ time:
  - **Reachability query:** return TRUE if and only if (a_n, b_m) is a reachable position w.r.t. σ.
  - **A single change in σ:** switch $\sigma(a_i, b_j)$ from 1 to 0 or vice versa.

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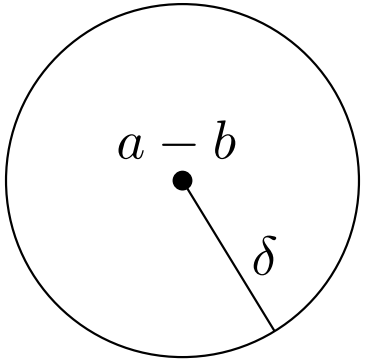

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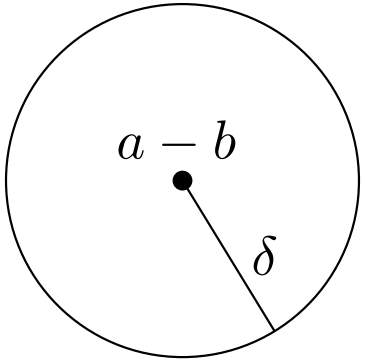

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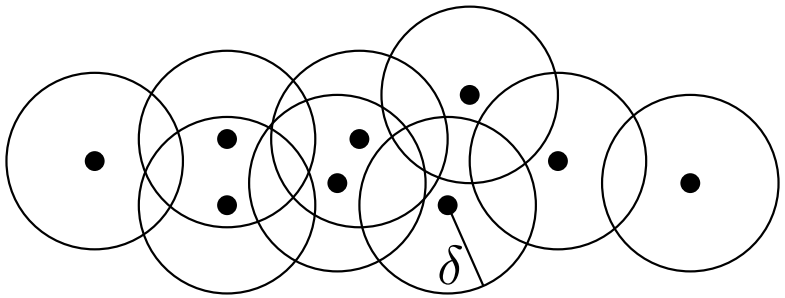

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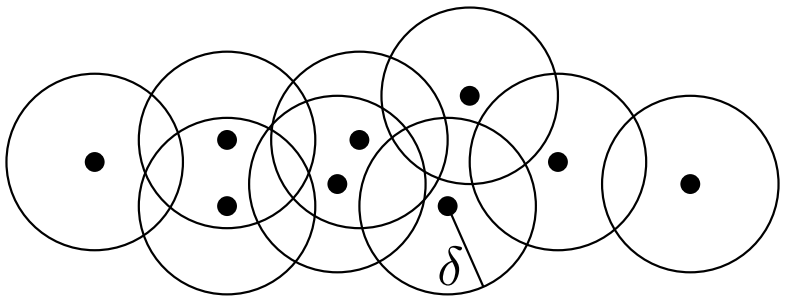

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STEP 3: Use **parametric search** in order to find an optimal translation (adds only a $O(\log(m + n))$ factor to the running time).

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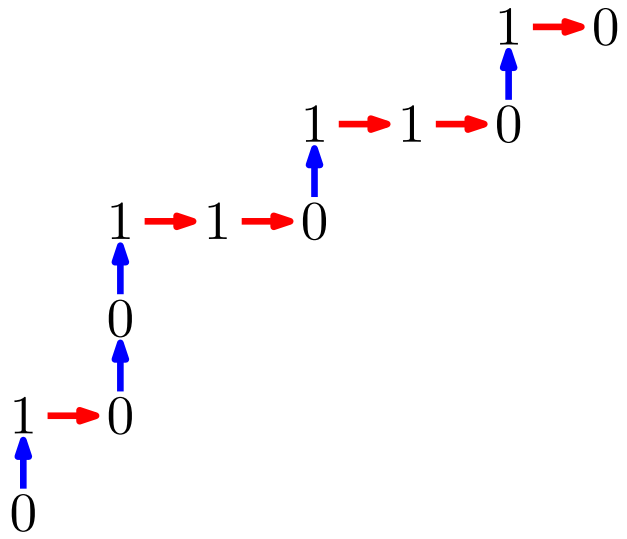
Build a graph of s-paths!

(partial) s-paths

A path P in G_δ from (a_i, b_j) to $(a_{i'}, b_{j'})$, $i \leq i'$, $j \leq j'$, is a **partial s-path**, if for each b_l, $j \leq l < j'$, there exists a position $(a_k, b_l) \in P$ that is valid (i.e., $\sigma_\delta(a_k, b_l) = 1$).



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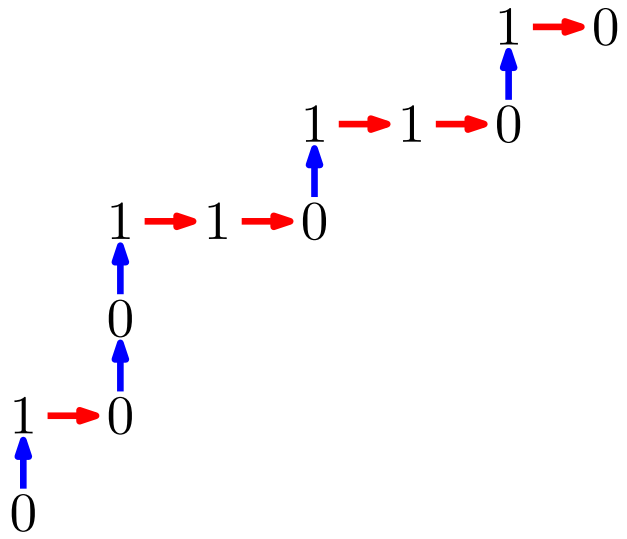

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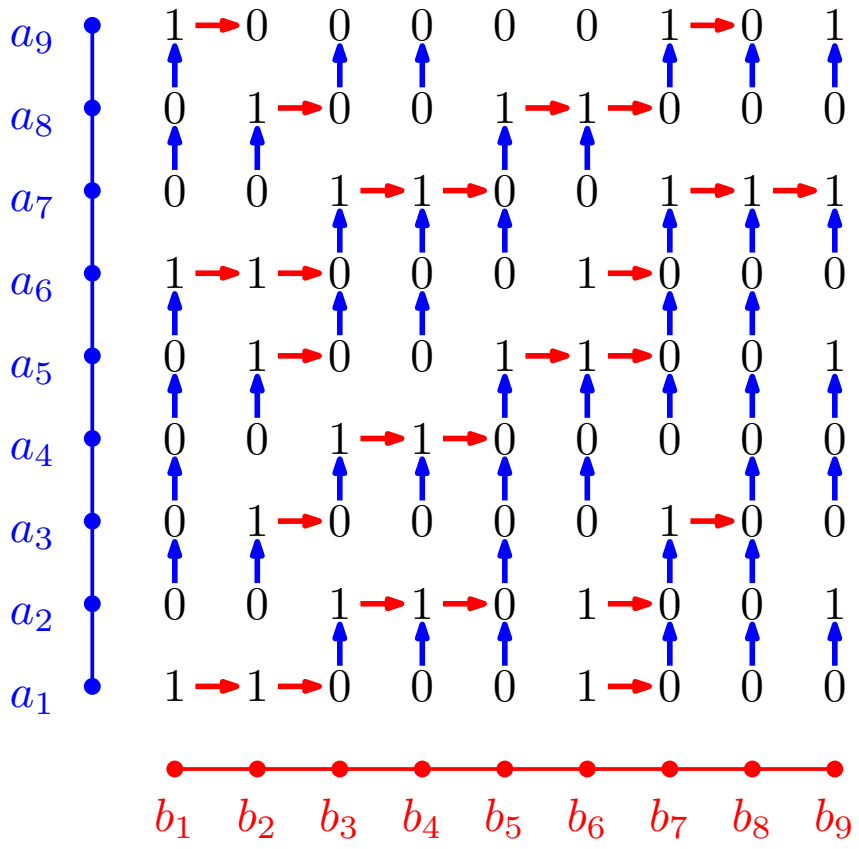
The graph of (partial) s-paths

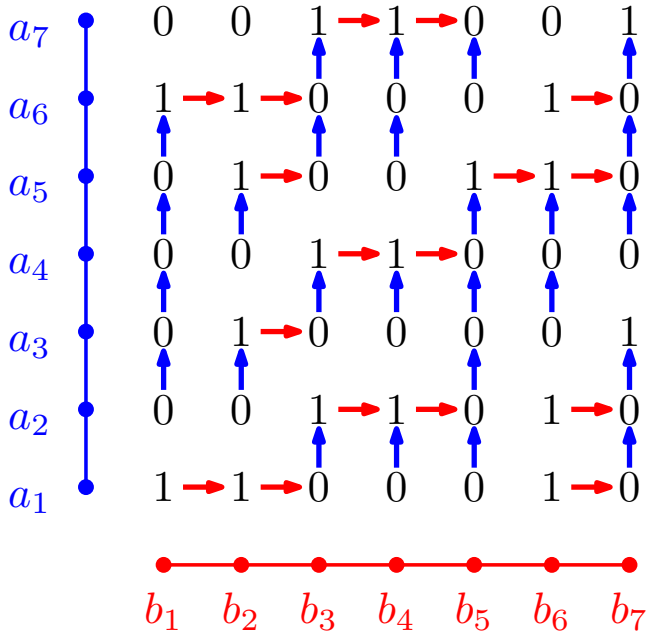
Consider $G_\delta = G(V = A \times B, E = E'_A \cup E'_B)$

$$E'_A = \{\langle (a_i, b_j), (a_{i+1}, b_j) \rangle \mid \sigma_\delta(a_i, b_j) = 0\}$$

$$E'_B = \{\langle (a_i, b_j), (a_i, b_{j+1}) \rangle \mid \sigma_\delta(a_i, b_j) = 1\}$$



Properties of G_δ



Property 1

All the paths in G_δ are partial s-paths.

Property 2

G_δ is a set of rooted binary trees
(where the root is a vertex of out-degree 0).

Lemma

(a_n, b_m) is an s-reachable position in G w.r.t. σ_δ, if and only if

- $\sigma_\delta(a_1, b_1) = 1$ *and* $\sigma_\delta(a_n, b_m) = 1$.
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$\Rightarrow$: Clearly, $\sigma_\delta(a_1, b_1) = 1$ and $\sigma_\delta(a_n, b_m) = 1$,

Let P be an s-path in G from (a_1, b_1) to (a_n, b_m).

Let P' be the path in G_δ from (a_1, b_1) to its root.

P' is always not above P: if a position (a_i, b_j) is an s-reachable position in G, then there exists a position $(a_{i'}, b_j) \in P'$, $i' \leq i$, such that $\sigma_\delta(a_{i'}, b_j) = 1$.

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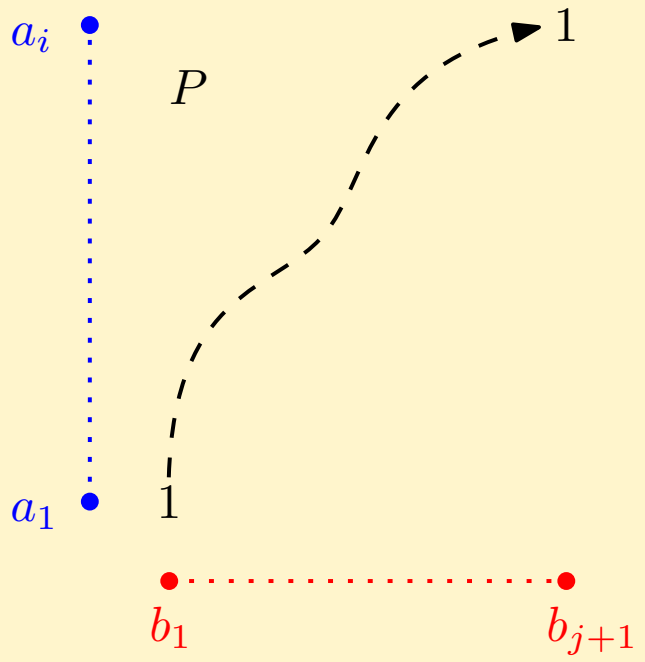

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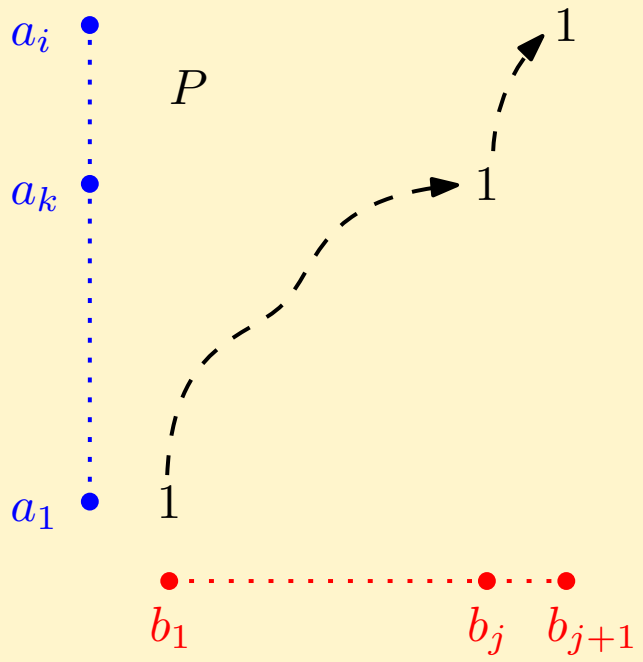

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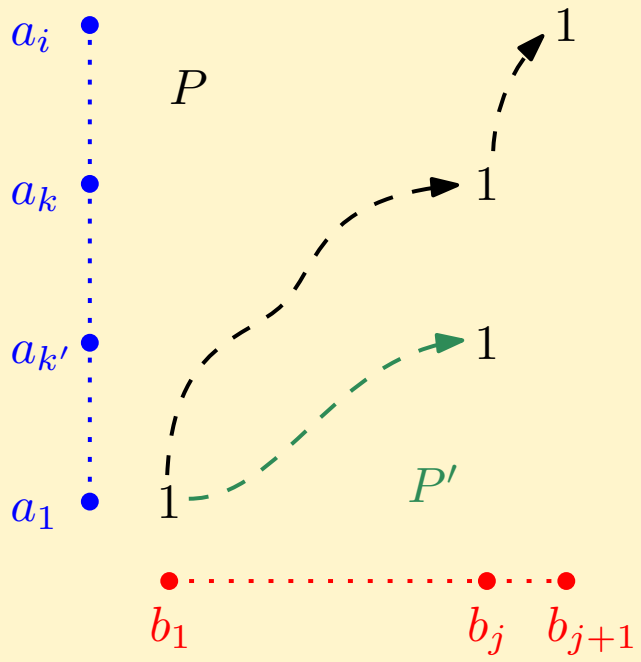

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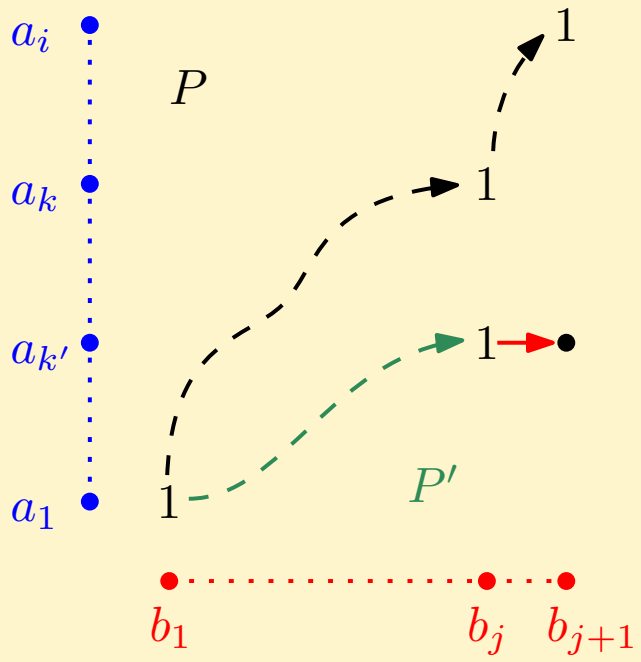

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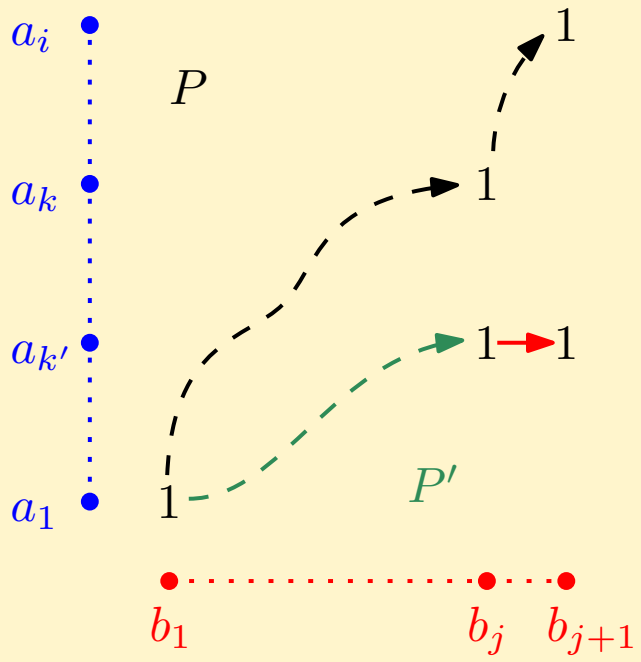

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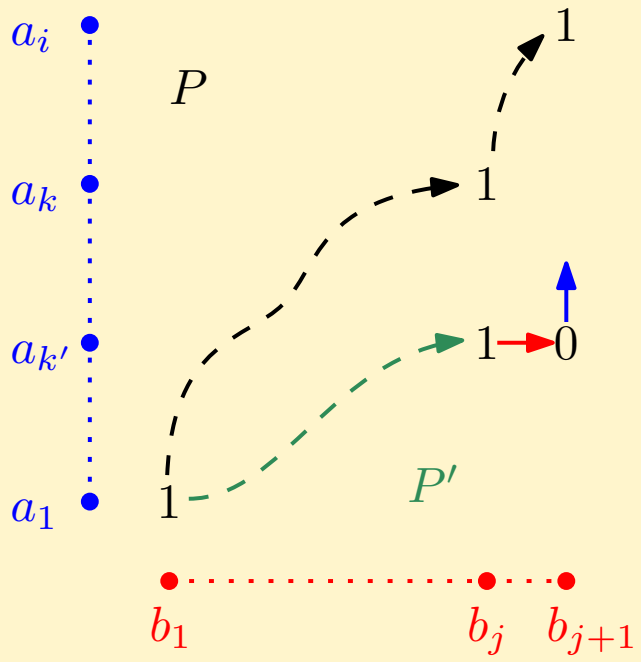

A Link-Cut tree

We represent G_δ using the Link-Cut tree data structure:

Sleator and Tarjan ('83)

The Link-Cut tree data structure stores a set of rooted trees and supports the following operations in $O(\log n)$ amortized time:

- $Link(v, u)$ — connect a root v to another node u as its child.
- $Cut(v)$ — disconnect the subtree rooted at v from its tree.
- $FindRoot(v)$ — find the root of the tree to which v belongs.

The dynamic data structure for DFDS

- Switch $\sigma_\delta(a_i, b_j)$ from 1 to 0:
  - remove the edge $\langle (a_i, b_j), (a_i, b_{j+1}) \rangle$ ($Cut(a_i, b_j)$).
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Theorem

Given sequences A and B with n and m points respectively in the plane, $\widehat{d^s_{dF}}(A, B)$ can be computed in $O(m^2 n^2 \log^2(m+n))$-time.

Back to the algorithm of BKS

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A more direct approach for translation in $\mathbb{R}^d$

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- S is the smallest sphere for which (a_n, b_m) is a reachable position w.r.t. $\sigma_S \Rightarrow \boxed{\delta' \geq \delta}$.

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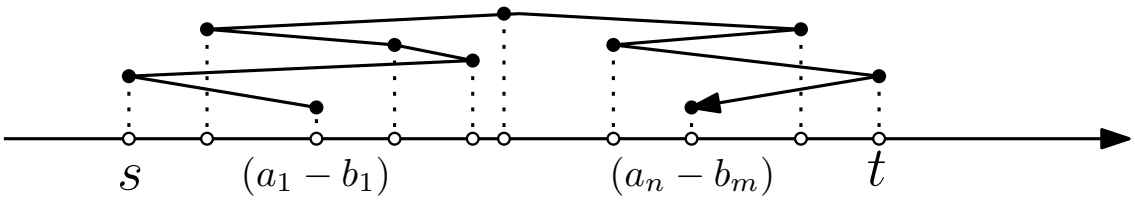
New indicator function: given a range $[s, t]$,

$$\sigma_{[s,t]}(a, b) = \begin{cases} 1, & s \leq a - b \leq t \\ 0, & \text{otherwise} \end{cases}$$



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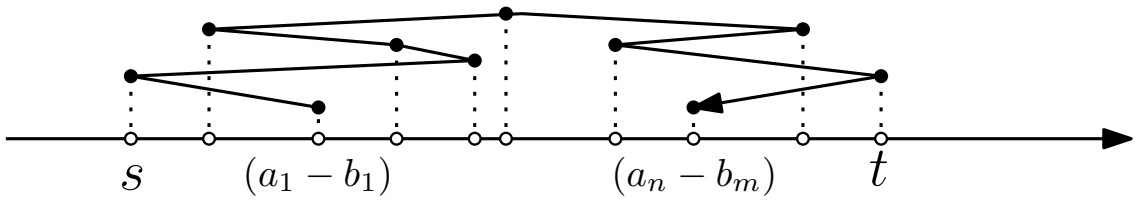

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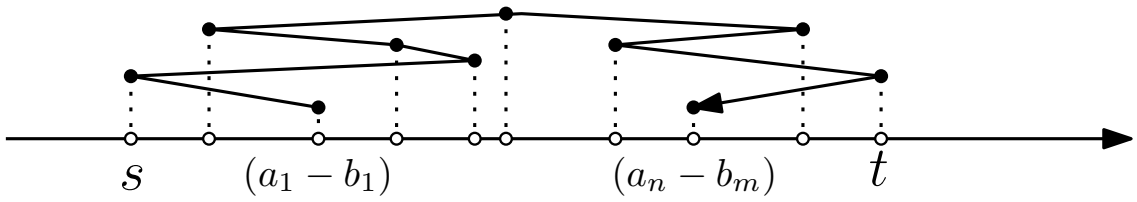

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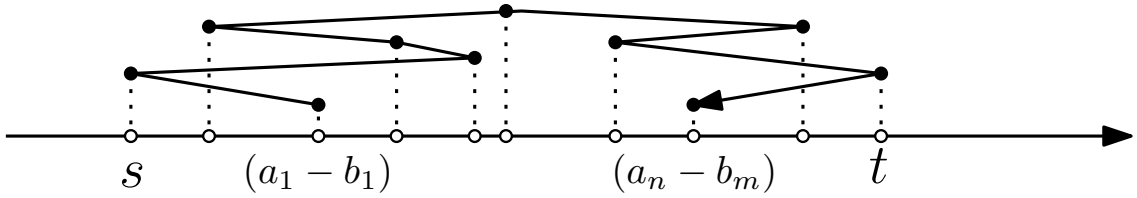
New indicator function: given a range $[s, t]$,

$$\sigma_{[s,t]}(a, b) = \begin{cases} 1, & s \leq a - b \leq t \\ 0, & \text{otherwise} \end{cases}$$



A range $[s, t]$ is a **feasible range** if (a_n, b_m) is a reachable position in G w.r.t $\sigma_{[s,t]}$ (there exists a path P from (a_1, b_1) to (a_n, b_m), such that $s \leq \max\{a - b \mid (a, b) \in P\} \leq \min\{a - b \mid (a, b) \in P\} \leq t$).

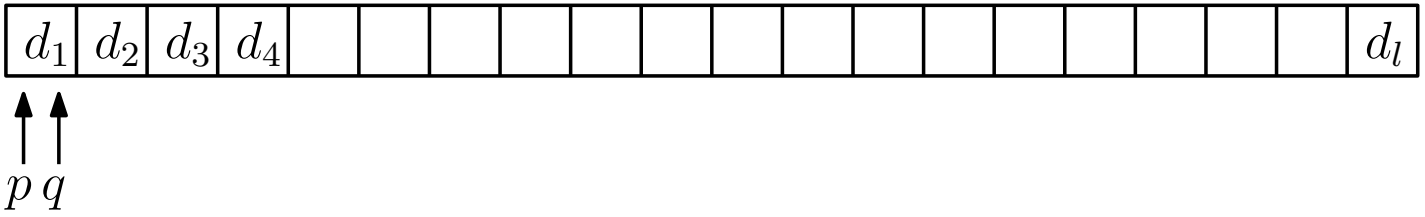
Let $\mathcal{D} = \{a_i - b_j \mid a_i \in A, b_j \in B\}$.

New goal: Find the smallest feasible range delimited by two points of $\mathcal{D}$.

Algorithm:

- Sort the values in $\mathcal{D} = \{d_1, \ldots, d_l\}$ such that $d_1 < d_2 < \cdots < d_l$, where $l = mn$.

$[d_p, d_q]$ is not a feasible range



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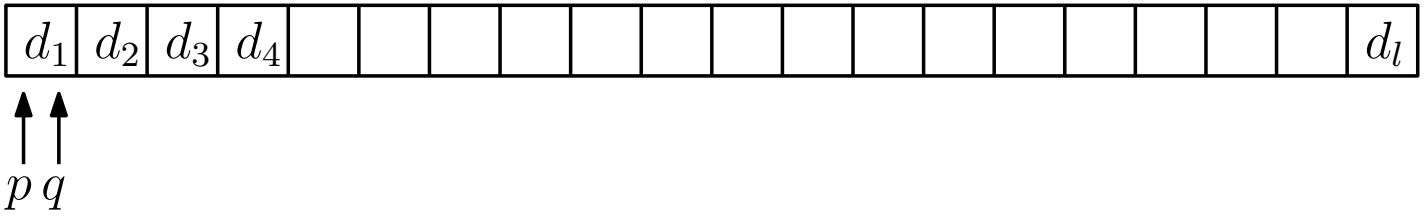

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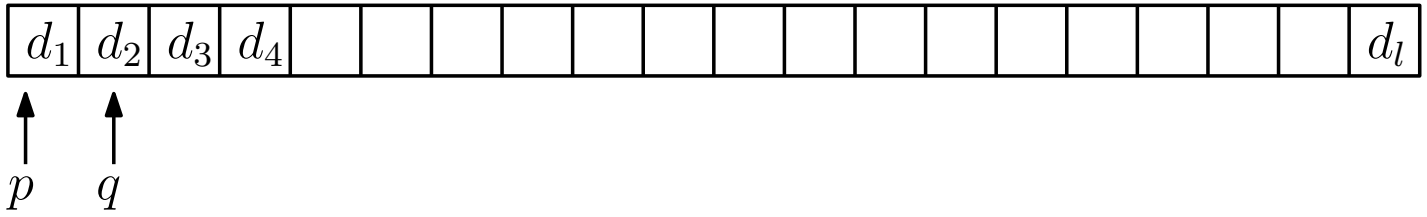

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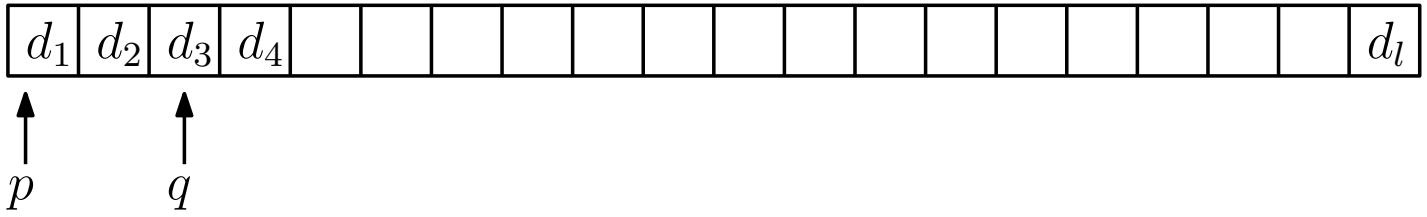

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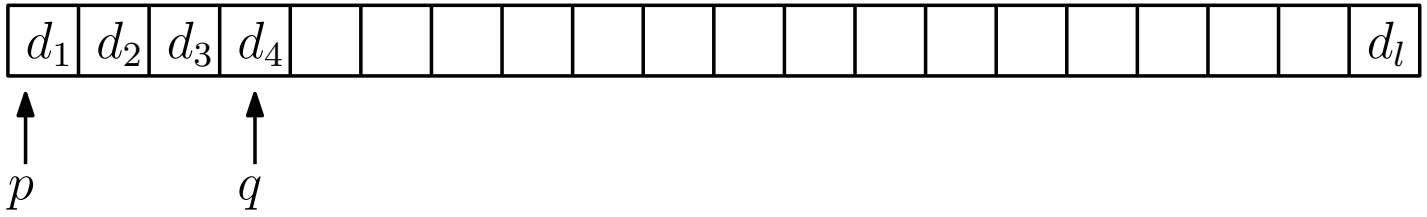

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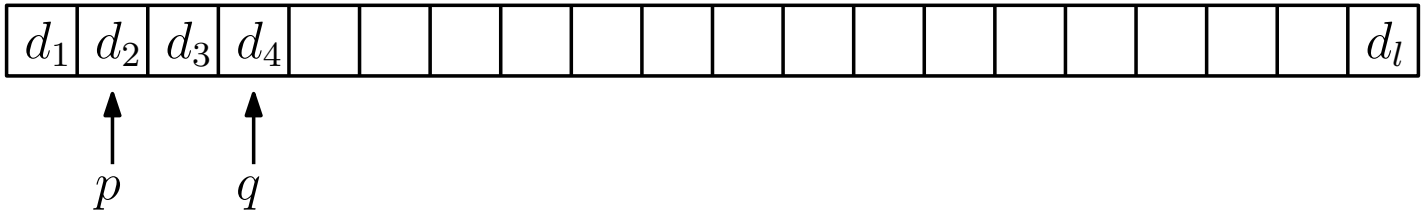

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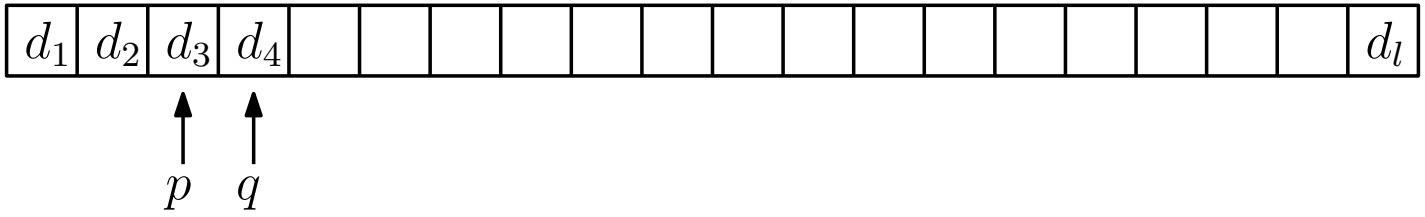

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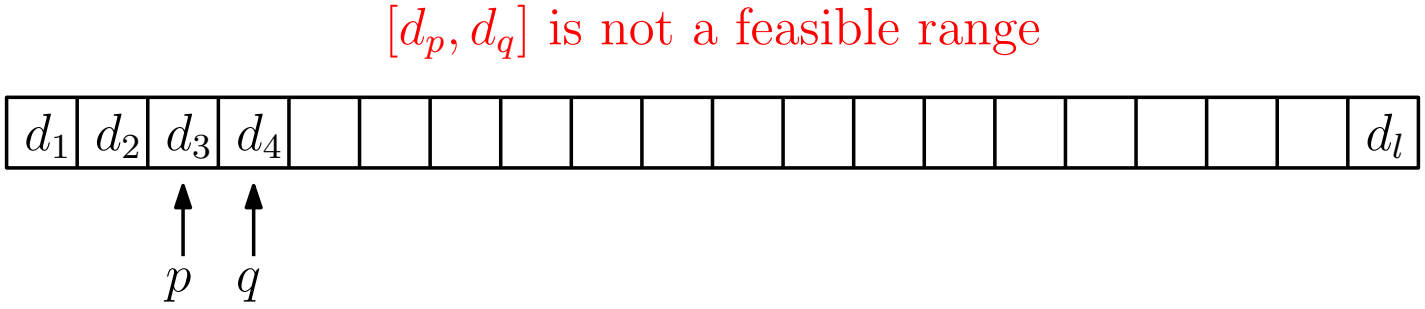

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Theorem

Let A and B be two sequences of n and m points ($m \leq n$), respectively, on the line. Then, $\widehat{d_{dF}}(A, B)$ can be computed in $O(m^2 n(1 + \log(n/m))$ time, and $\widehat{d^s_{dF}}(A, B)$ and $\widehat{d^w_{dF}}(A, B)$ can be computed in $O(mn \log(m + n))$ time.

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Goal: find the smallest feasible range.

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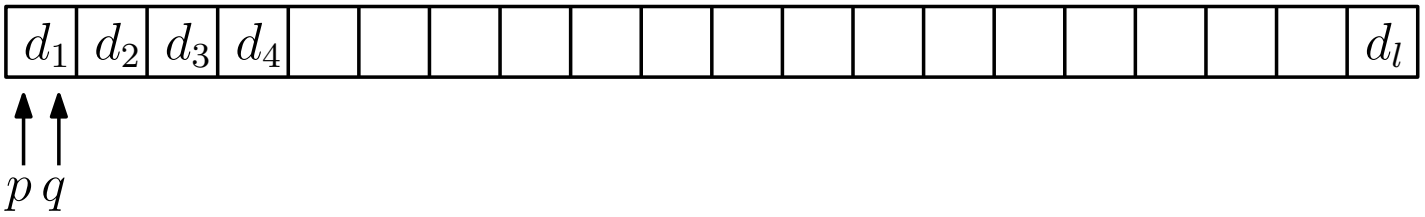

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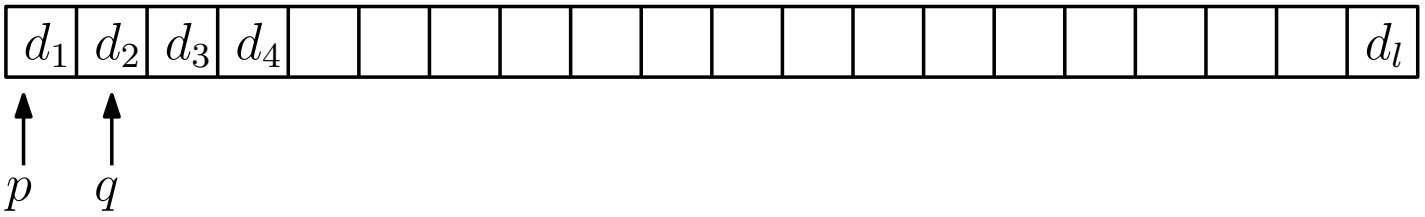

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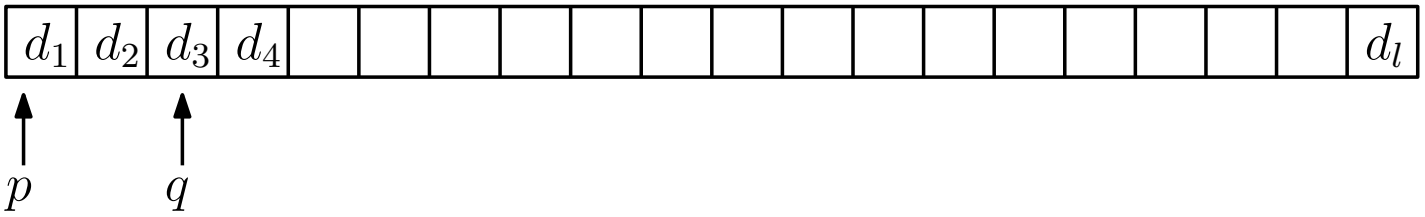

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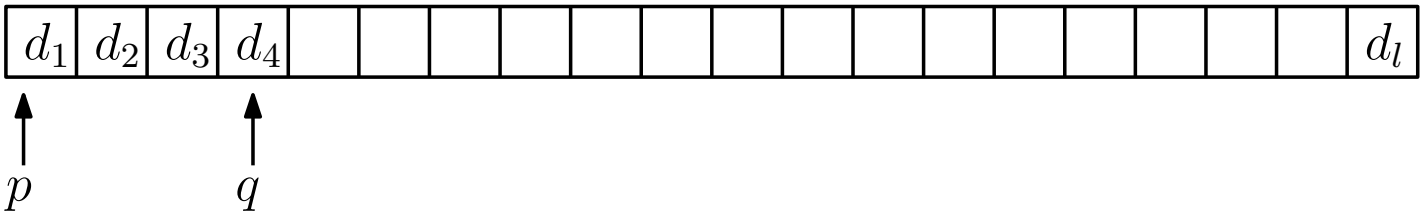

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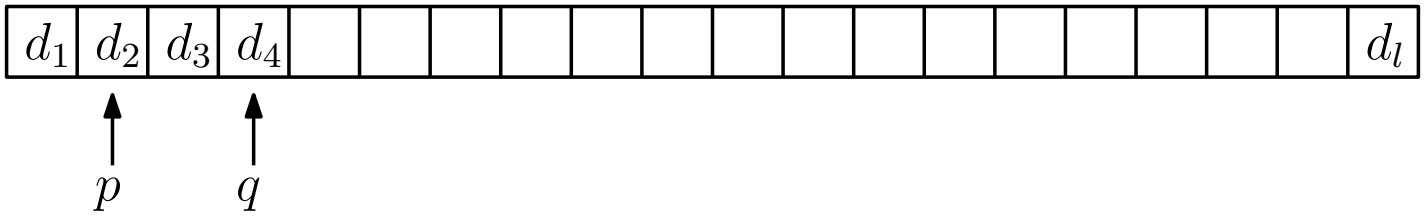

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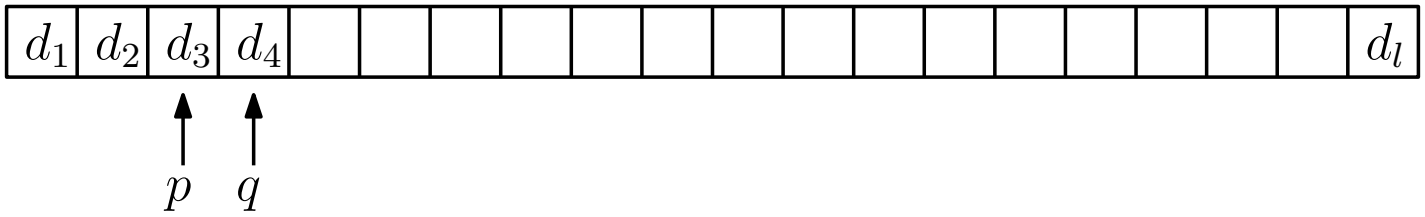

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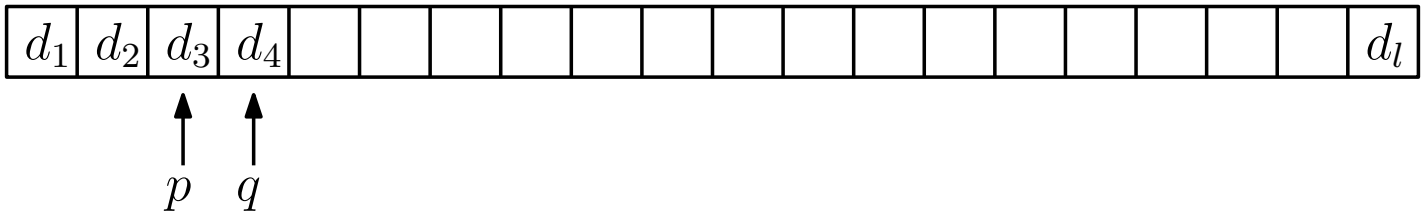

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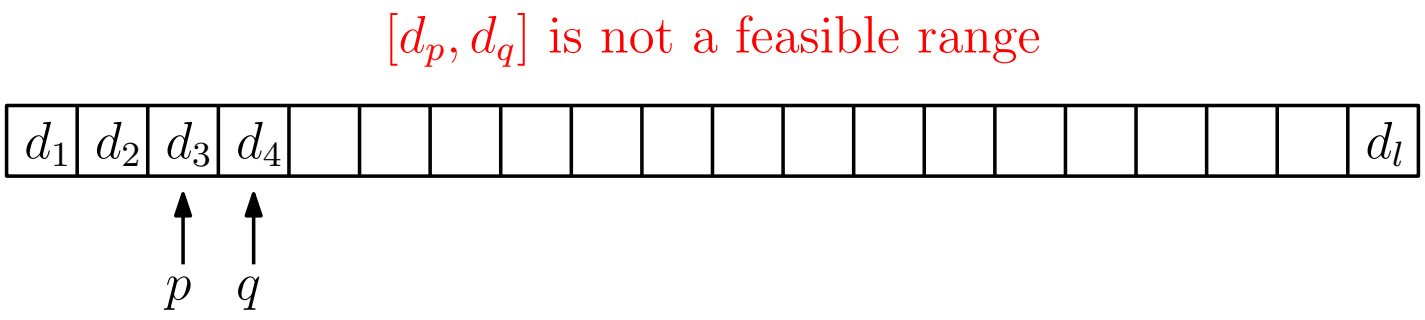

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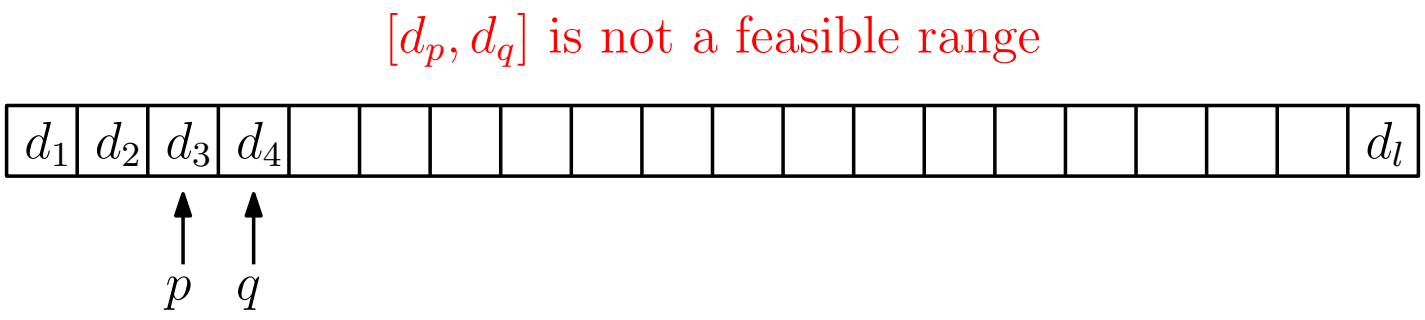
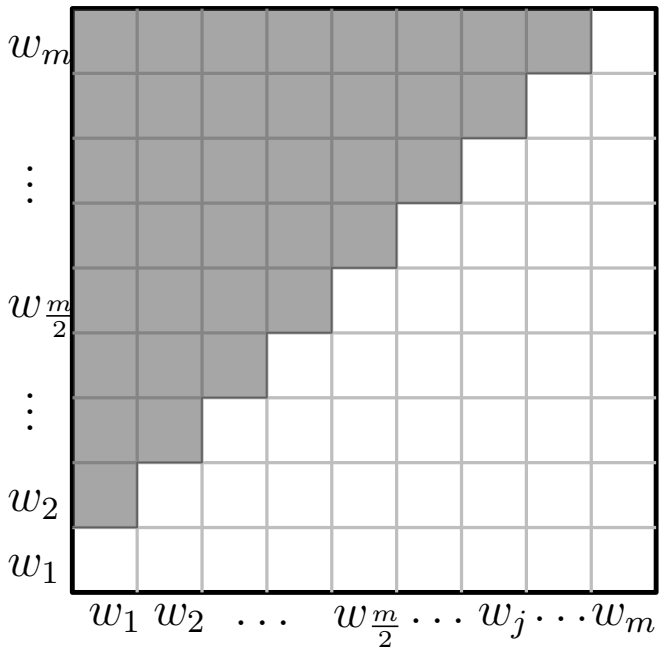
  - Especially useful when an efficient **dynamic version** of the feasibility decider is available.
- We present an **alternative scheme for BOP** - does not require a dynamic version of the feasibility decider.

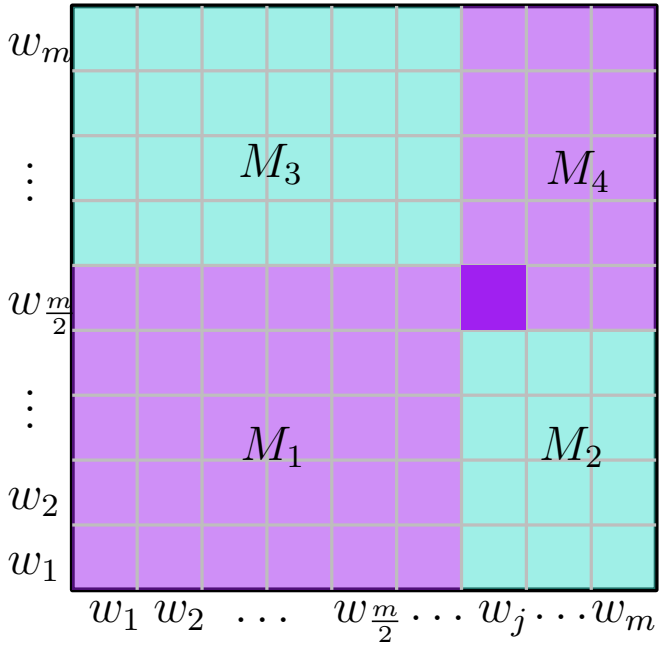


The matrix of ranges

- Let $w_1 = w(e_1) < w_2 = w(e_2) < \cdots < w_m = w(e_m)$.
- Let M be the matrix whose rows and columns correspond to $w_1, w_2, \ldots, w_m$.
- Cell $M_{i,j} \equiv$ Range $[w_i, w_j]$.



- $M_{i,j}$ is contained in all the ranges $M_{i',j'}$ with $i' \leq i \leq j \leq j'$.



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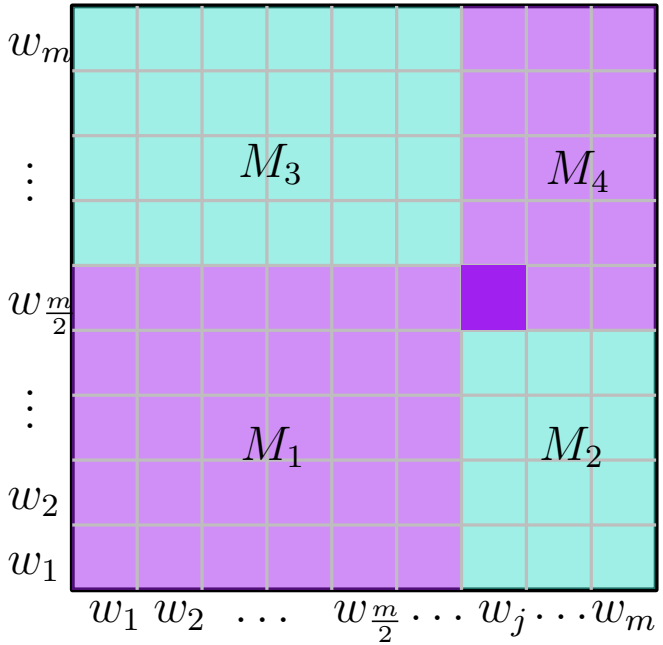

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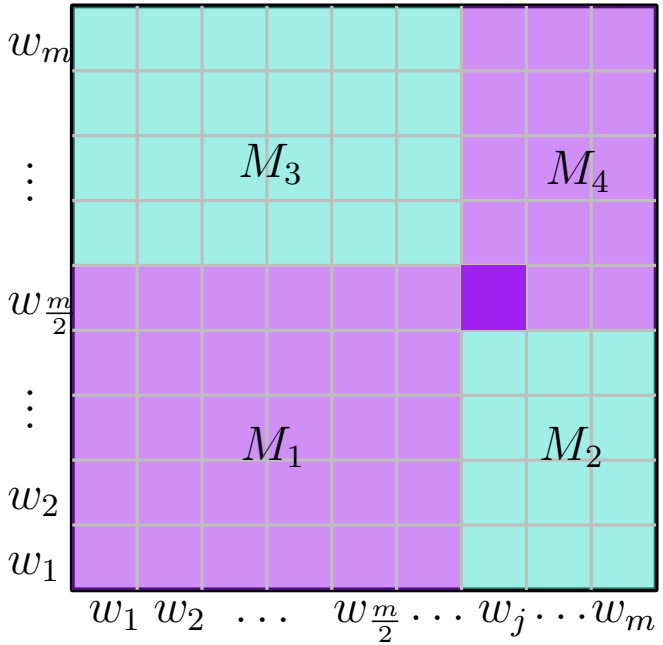
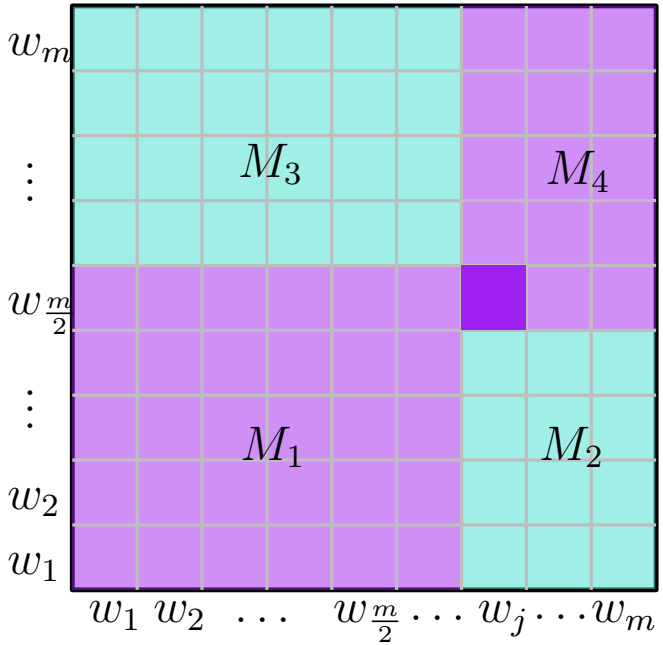
- $M_{\frac{m}{2},j}$ induces a partition of M into 4 submatrices: M_1, M_2, M_3, M_4



- None of the ranges in M_1 is a feasible range.



- None of the ranges in M_1 is a feasible range.
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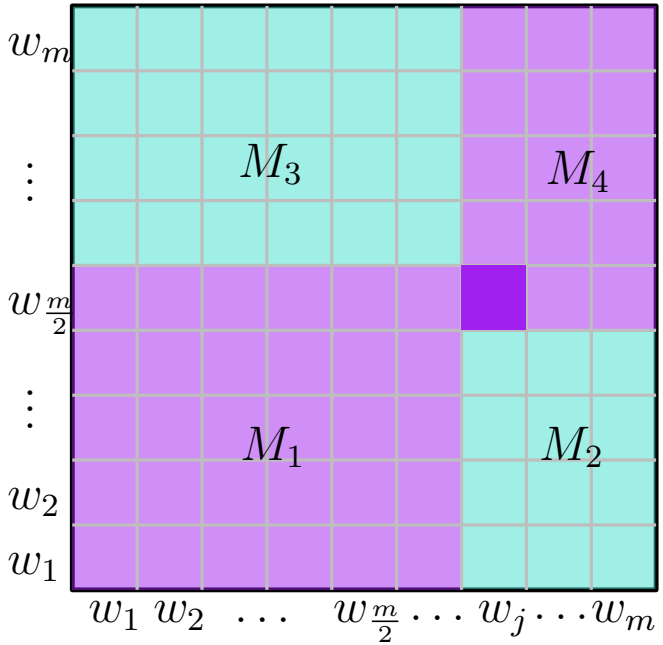

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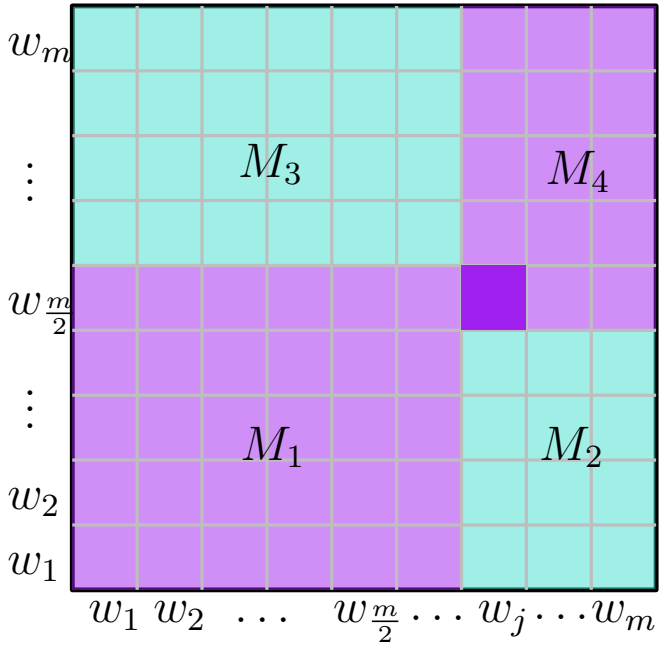

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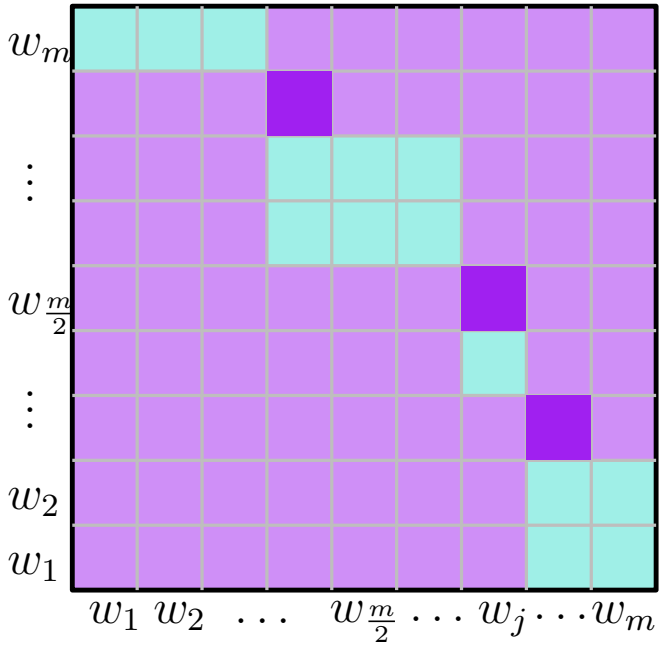

The algorithm

Each recursive call is associated with:

- a submatrix $M' = M([p, p'] \times [q', q])$ of M.
- a corresponding graph $G' = G([p, p'] \times [q', q])$.

A recursive call has 2 steps:

1. **Perform a binary search in the middle row** of M' to find the smallest feasible range in this row, using the corresponding graph G'.
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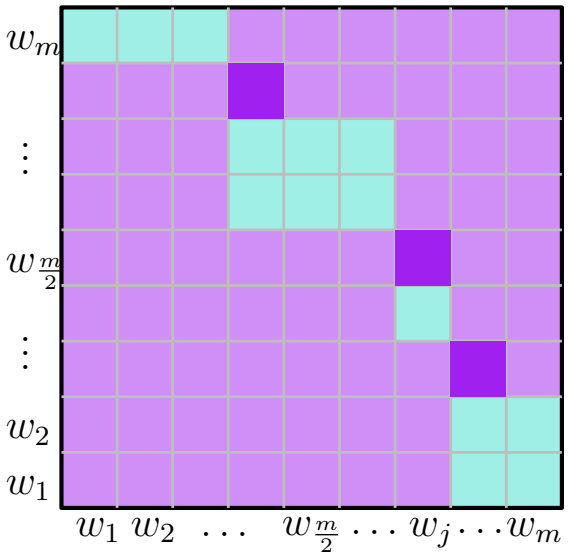
The scheme requires the followings properties of G':

1. The size of G' is $O(|M'|)$.
2. Given G', the feasibility decider can answer a feasibility query for any range in M', in $O(f(|G'|))$ time.
3. Constructing the graphs for the next level takes $O(|G'|)$ time.

Running time

The recursion tree consists of $O(\log m)$ levels.

- The i'th level is associated with 2^i disjoint submatrices of M.
- In the i'th level we apply the recursive algorithm to each of the 2^i submatrices associated with this level.



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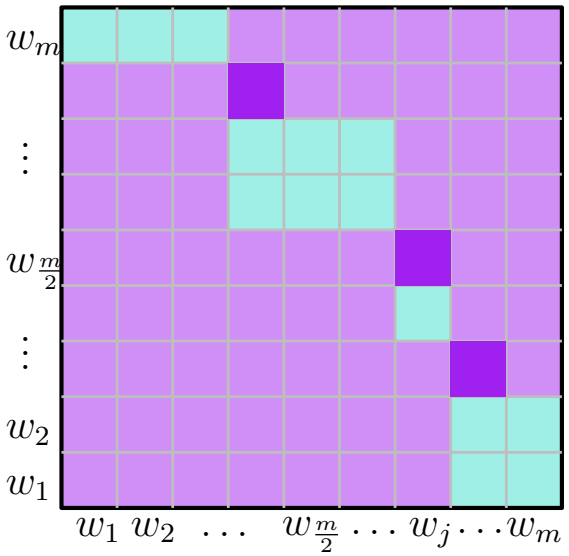

- Let $\{M_k^i\}_{k=1}^{2^i}$ be the submatrices associated with the i'th level.
- Let G_k^i be the graph corresponding to M_k^i.

In each recursive call:

1. The size of G_k^i is $O(M_k^i)$.
2. The feasibility decider runs in $O(f(|G_k^i|))$ time $\Rightarrow$ the binary search in M_k^i runs in $O(f(|M_k^i|)\log|M_k^i|)$ time.
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- The total time spent on the i'th level is $O(\sum_{k=1}^{2^i}(|M_k^i| + f(|M_k^i|)\log|M_k^i|))$

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Lemma

The total size of the matrices in each level is at most $2m$.

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if $f(|M_k^i|) = O(|M_k^i|)$, then we get $O(m\log^2 n)$.

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$G = (V, E, w)$ is a weighted graph with n vertices and m edges.

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Given two vertices $s, t \in V$, find a path P^* in G between s and t, which minimizes $P_{max} - P_{min}$, over all paths P between s and t.

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- Our scheme: simpler algorithm with the same running time.

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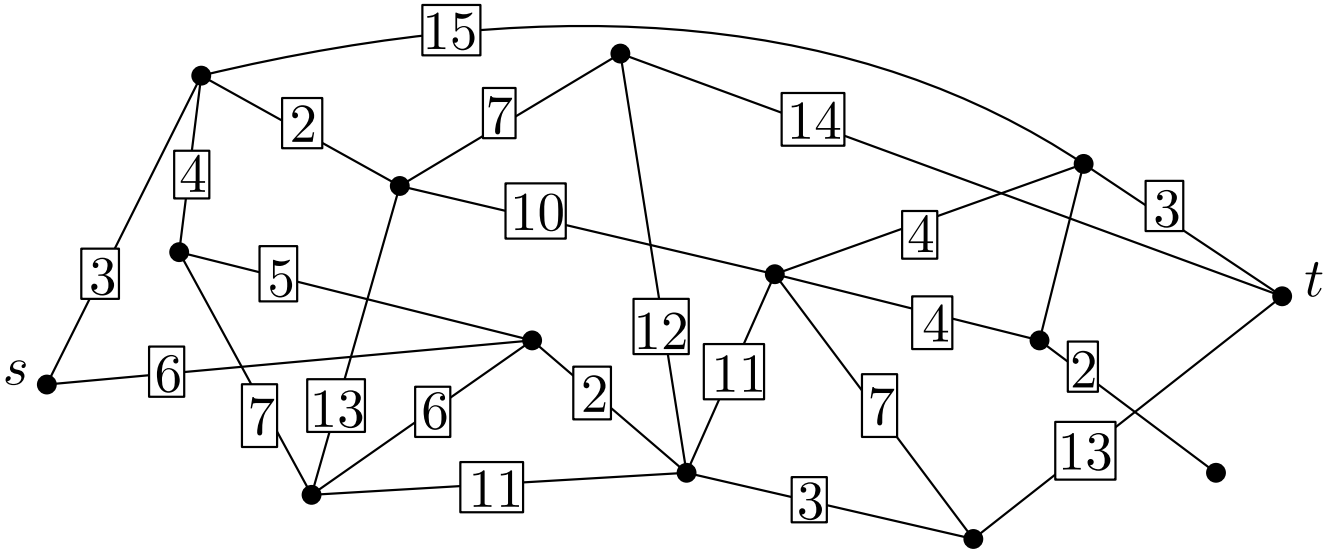

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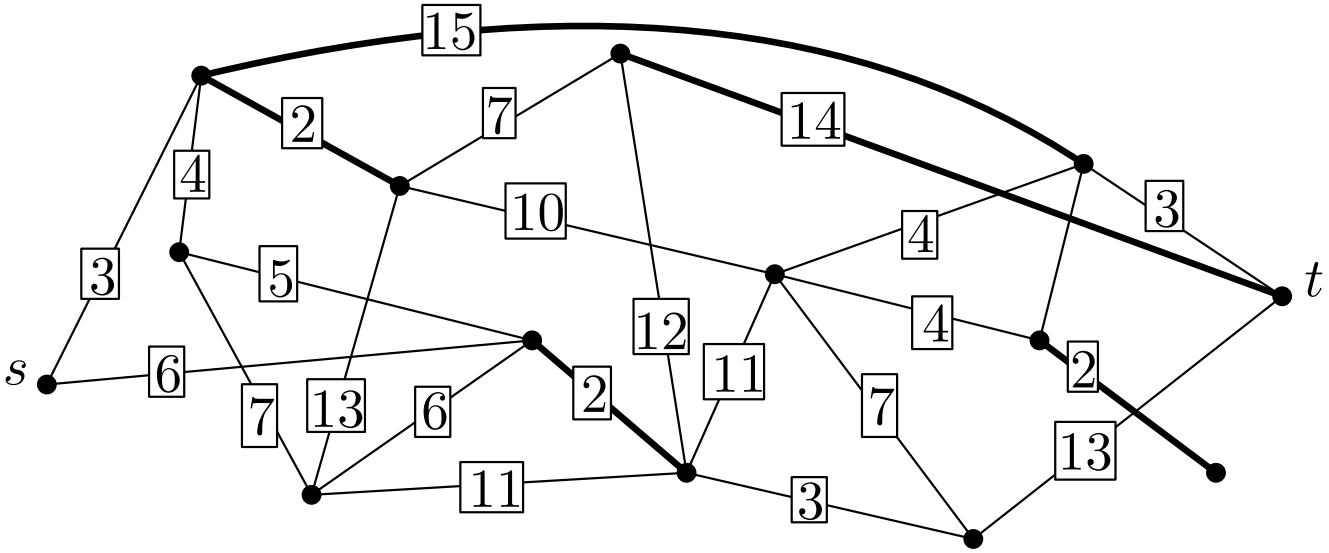

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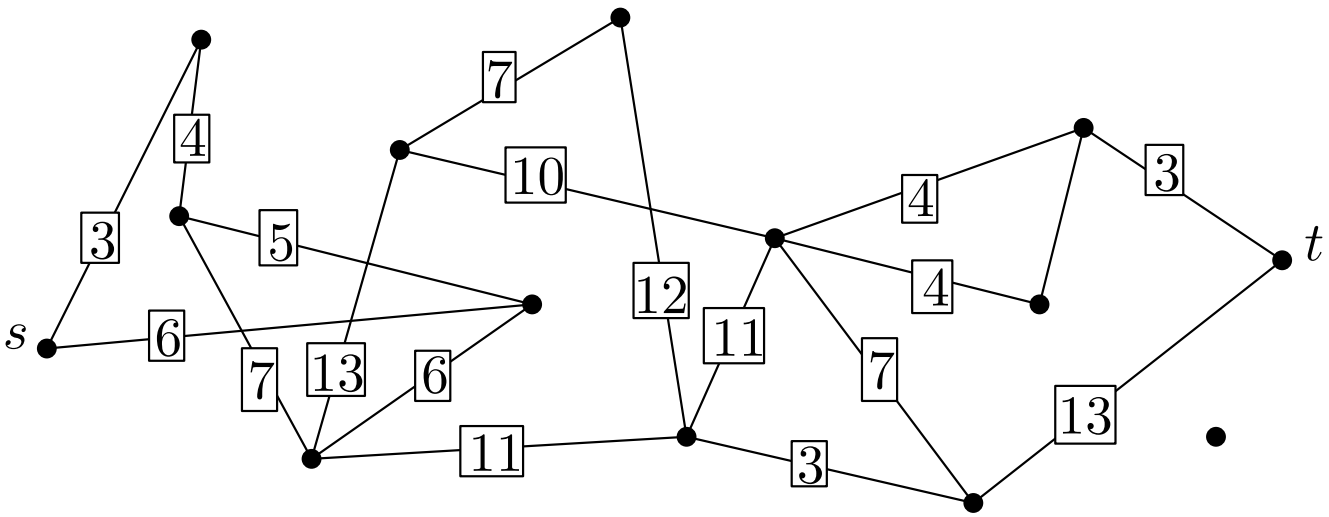

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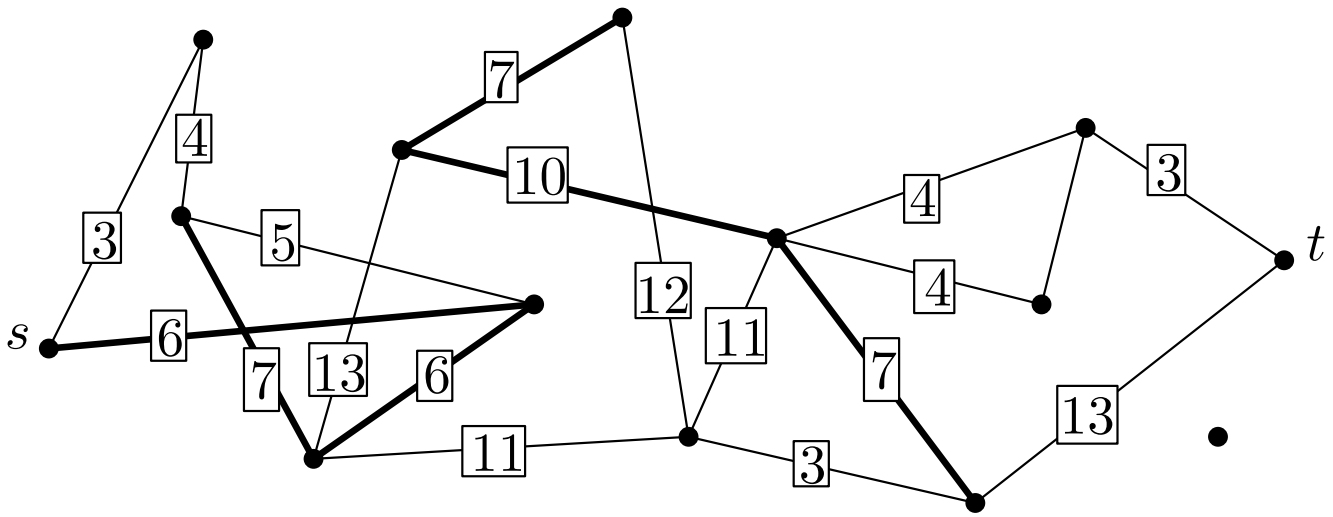

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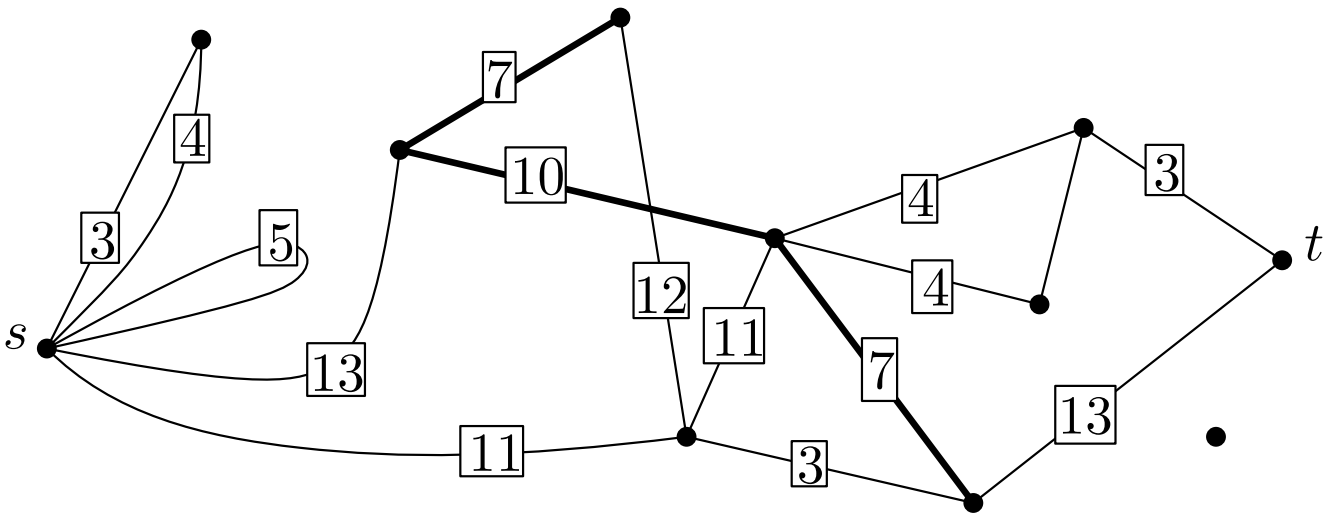

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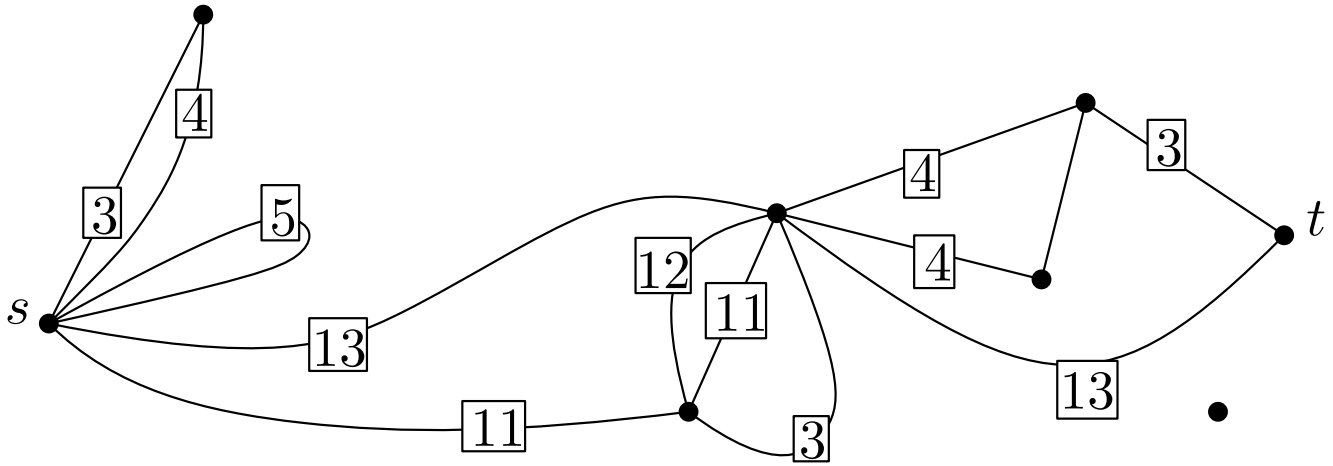

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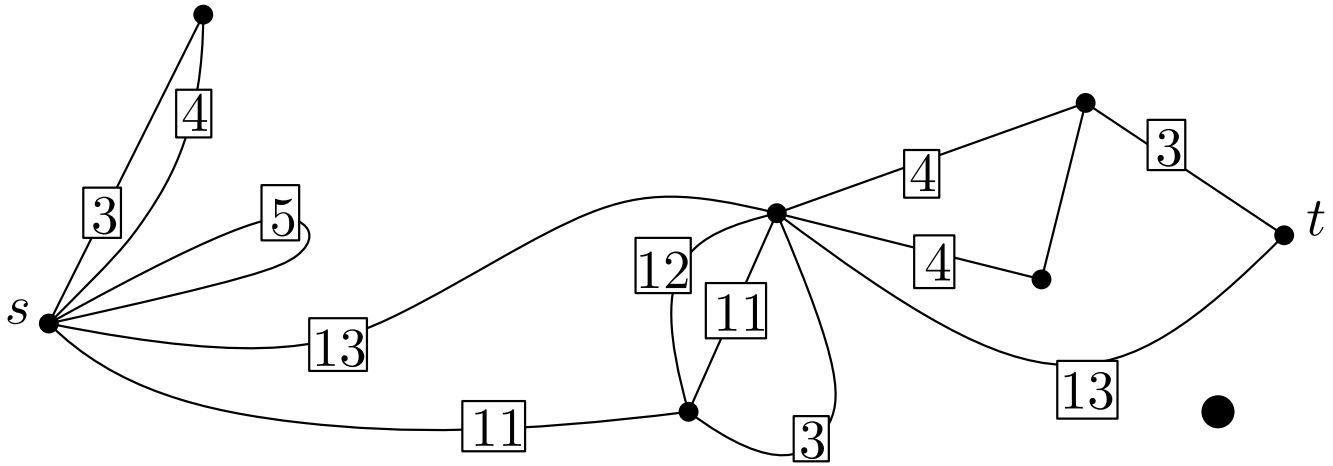

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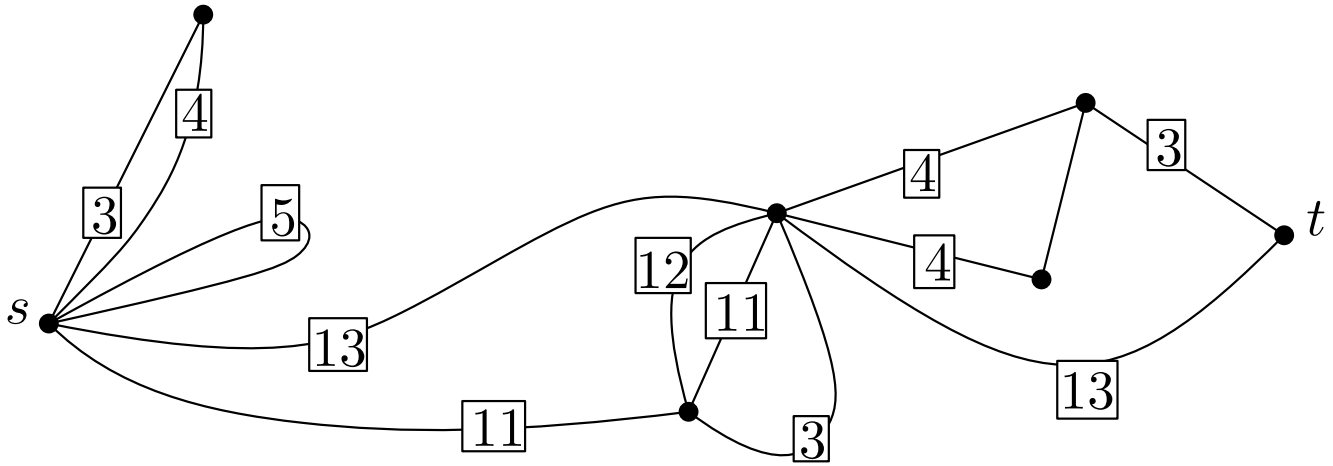

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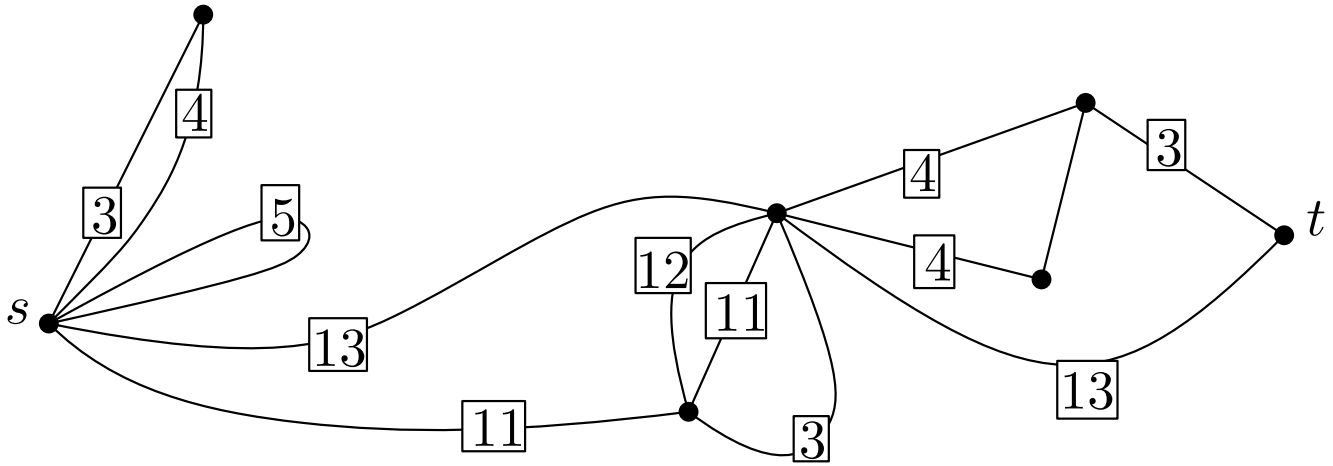

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Running time: $O(|G'|)$.

WDFD under translation in 1D

WDFD under translation in 1D can be viewed as BOP:
$[s, t]$ is a **feasible range** $\Leftrightarrow$
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WDFD under translation in 1D is a special case of MUPP!

MUPP $\Rightarrow$ WDFD under translation in 1D

For MUPP we need a **weighted graph** $\tilde{G}_w = (\tilde{V}_w, \tilde{E}_w, \omega)$
$\tilde{V}_w = (A \times B) \cup \{v_e \mid e \in E_w\}$, $\tilde{E}_w = \{(u, v_e), (v_e, v) \mid e = (u, v) \in E_w\}$,
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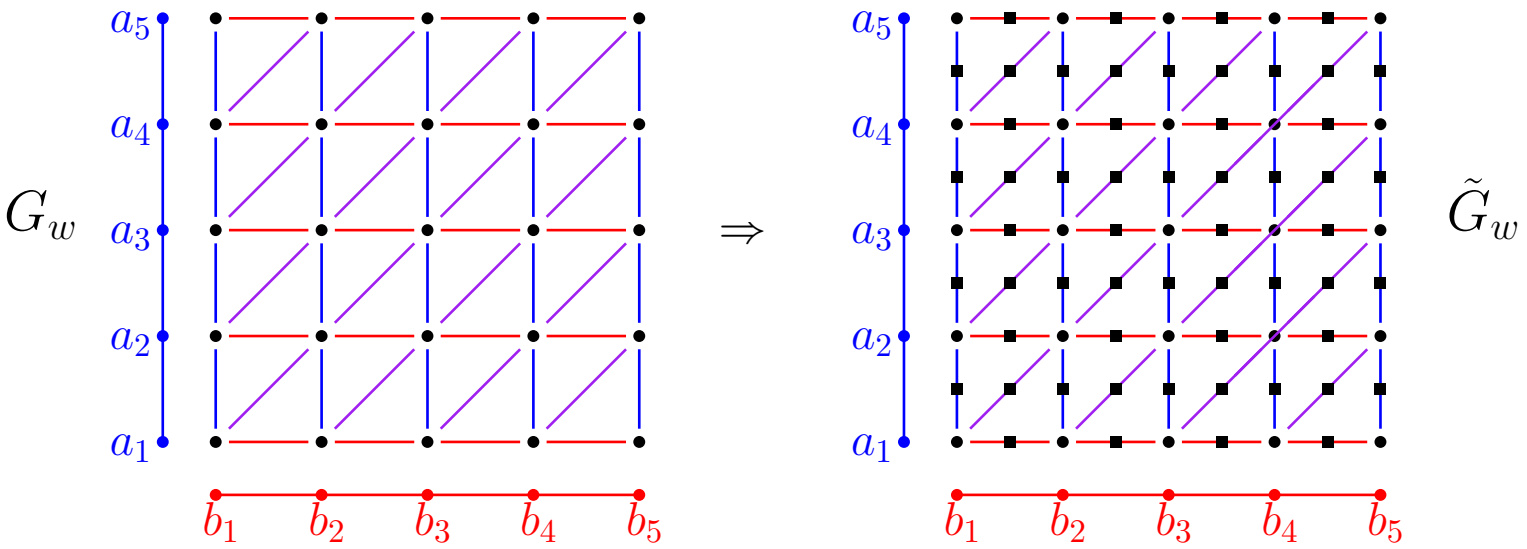

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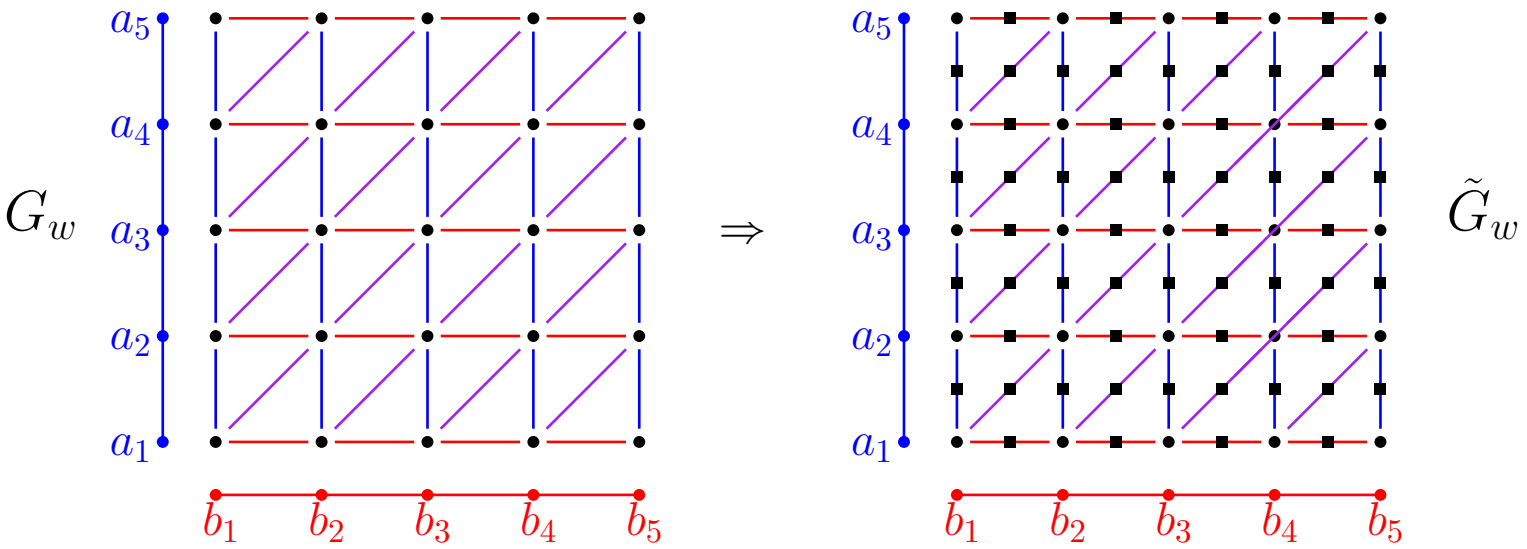

(a_n, b_m) is a **w-reachable position** in G_w w.r.t. $\sigma_{[s,t]} \Leftrightarrow$
there exists a path $\tilde{P}$ between (a_1, b_1) and (a_n, b_m) in $\tilde{G}_w$ such that for each edge $e \in \tilde{P}$, $\omega(e) \in [s, t]$.

MUPP $\Rightarrow$ WDFD under translation in 1D

MUPP between (a_1, b_1) and (a_n, b_m) in $\tilde{G}_w \Rightarrow$
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Theorem

Let $A = (a_1, \ldots, a_n)$ and $B = (b_1, \ldots, b_m)$ be two sequences of points in 1D. Then, the weak discrete Fréchet distance under translation, $\widehat{d^w_{dF}}(A, B)$, can be computed in $O(mn \log^2(m+n))$ time.

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Our algorithm: slower by a factor of $\log n$, BUT does not require any special data structures, and has easy and shorter description.

A new variant: The discrete Fréchet gap

Two frogs, two curves: $A = (a_1, ..., a_n)$, $B = (b_1, ..., b_n)$.
Same rules: traverse all the points in order, no backtracking.

New indicator function: given a range $[s, t]$,

$$\sigma_{[s,t]}(a, b) = \begin{cases} 1, & s \leq d(a, b) \leq t \\ 0, & \text{otherwise} \end{cases}$$

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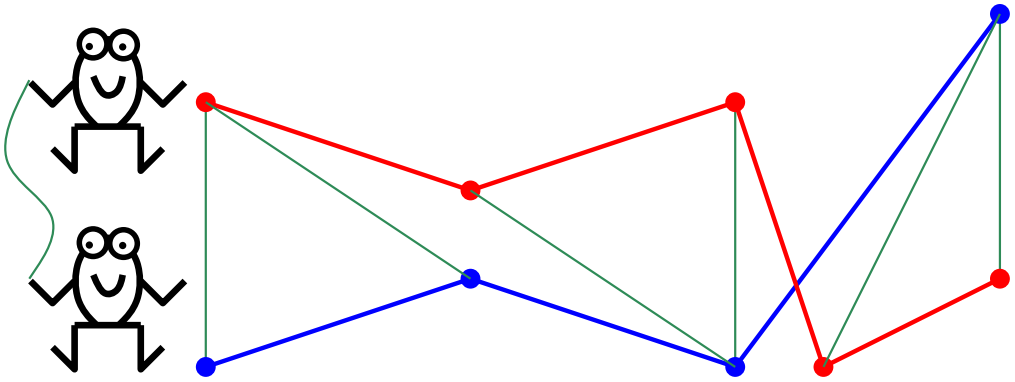
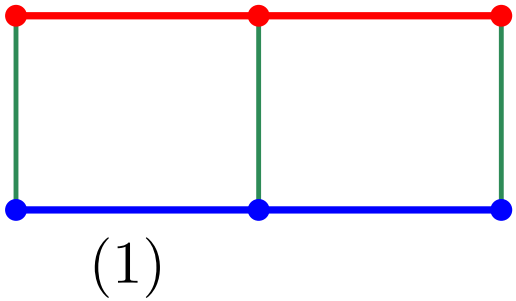
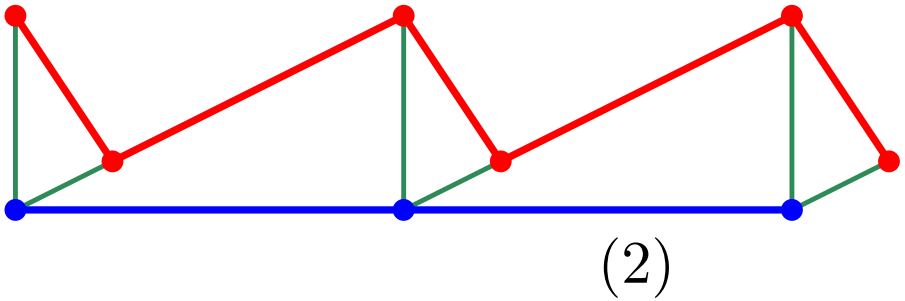
- The **discrete Fréchet gap** ($d^g_{dF}(A, B)$) is the smallest range $[s, t]$, $s \geq t \geq 0$, for which (a_n, b_m) is a reachable position w.r.t. $\sigma_{[s,t]}$.

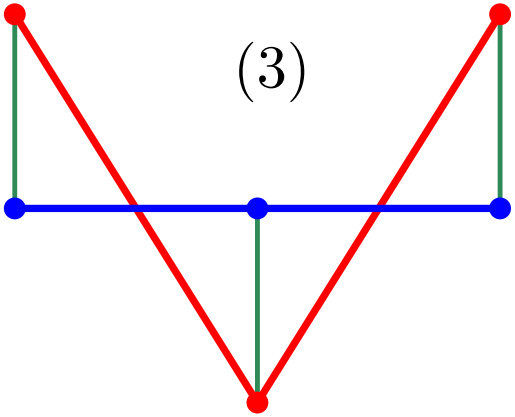


Is this a better variant?

- Gives a better reflection of reality?

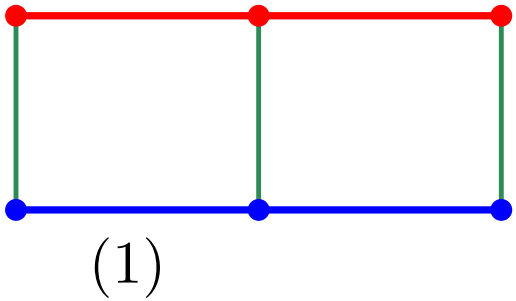


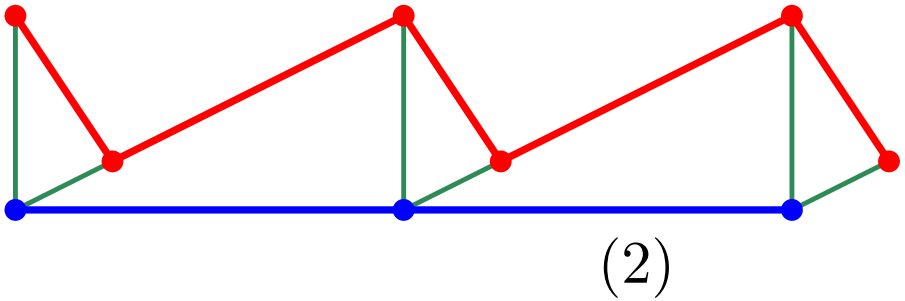


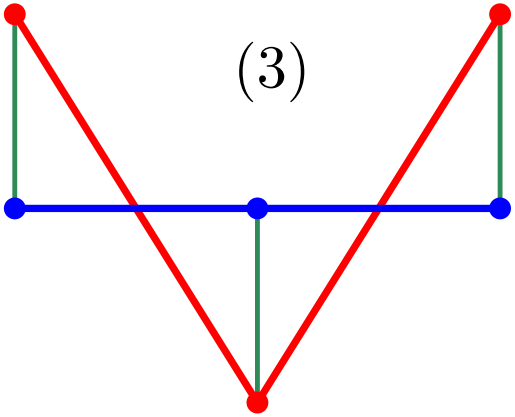


Is this a better variant?

- Gives a better reflection of reality? sometimes, but not always.

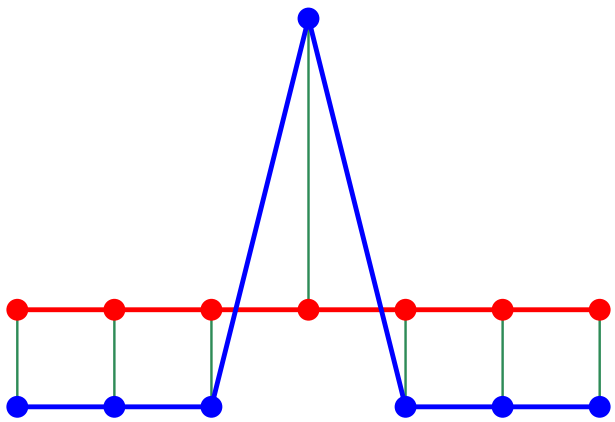






Is this a better variant?

- Gives a better reflection of reality? sometimes, but not always.
- Handling **outliers**? no.



Let us combine gap and shortcuts...

$A = (a_1, ..., a_n)$, $B = (b_1, ..., b_n)$, a leash of length δ.
Only the A-frog can skip points.

- The **(one-sided) discrete Fréchet gap with shortcuts** is the smallest range $[s, t]$, $s \geq t \geq 0$, for which (a_n, b_m) is an s-reachable position w.r.t. $\sigma_{[s,t]}$.

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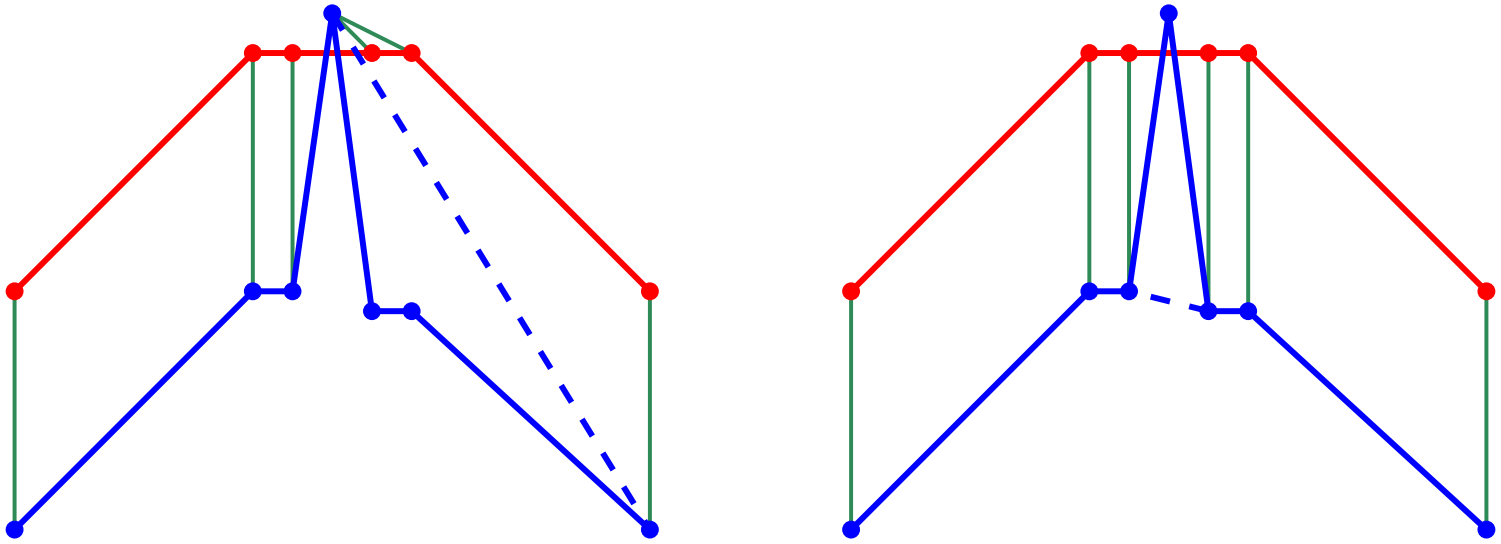
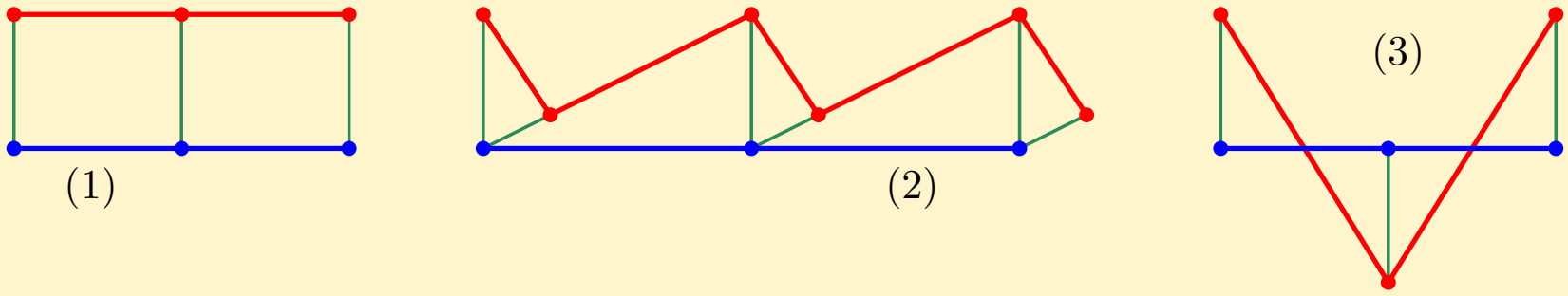


That seems to give better results!

What is between Gap and Translation?

Is there some connection between the discrete Fréchet gap and the discrete Fréchet distance under translation?



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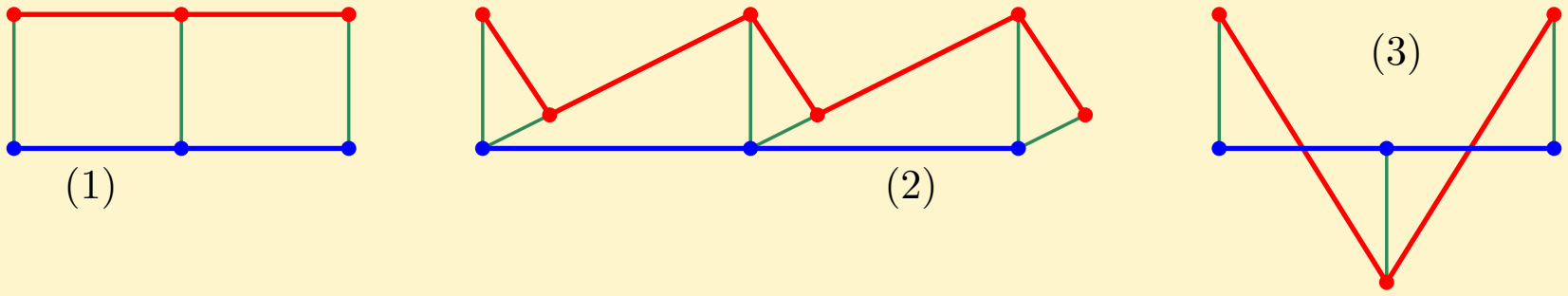

DFDS and WDFD, both in 1D under translation, are in some sense analogous to their respective gap variants (in d dimensions and no translation):

- We can use **similar algorithms** to compute them, but with **different indicator functions**!

Thank You!