Algorithms for the discrete Fréchet distance under translation

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Based on joint work with Matthew J. Katz
Similarity of curves

To what extent the two given curves resemble each other?
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**Applications:** Signature verification
Similarity of curves

To what extent the two given curves resemble each other?

Applications: Map-matching of vehicle tracking data
Similarity of curves

To what extent the two given curves resemble each other?

**Applications:** Analysis of moving objects
How to compare curves?

What distance measure between curves should be used?
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- Hausdorff distance.

\[ d_H = \max\{\sup_{a \in A} \inf_{b \in B} d(a, b), \sup_{b \in B} \inf_{a \in A} d(a, b)\} \]
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- **Hausdorff distance**.
  \[ d_H = \max\{ \sup_{a \in A} \inf_{b \in B} d(a, b), \sup_{b \in B} \inf_{a \in A} d(a, b) \} \]

  - Only takes into account the sets of points but not the ordering.
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✓ **Fréchet distance.**
The (continuous) Fréchet distance

A person and a dog connected by a leash of length $\delta$. They walk along the curves $A$ and $B$, respectively, no backtracking.
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- The **Fréchet distance** $(d_F(A, B))$ is the minimum $\delta$ that is sufficient for traversing both curves in this manner.
The discrete Fréchet distance

Two sequences of points, $A = (a_1, \ldots, a_n)$ and $B = (b_1, \ldots, b_m)$. Two frogs, the $A$-frog and the $B$-frog, connected by a leash of length $\delta$, hopping along their respective sequences.
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▶ The **discrete Fréchet distance** ($d_{dF}(A, B)$) is the **minimum** $\delta$ that allows the frogs to reach $a_n$ and $b_m$. 
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▶ The **discrete Fréchet distance** \( (d_{dF}(A, B)) \) is the minimum \( \delta \) that allows the frogs to reach \( a_n \) and \( b_m \).

▶ A good approximation of the continuous distance.
▶ Makes more sense in some situations (computational biology).
Related work

- **Eiter and Mannila (’94)** showed that $d_{dF}(A, B)$ can be computed in $O(n^2)$ time.

- **Agarwal et al. (’13)** showed how to compute $d_{dF}(A, B)$ in $O\left(\frac{n^2 \log \log n}{\log n}\right)$ time.

- **Bringmann and Mulzer (’15)** presented a conditional lower bound that strongly subquadratic algorithms for the discrete Fréchet distance are unlikely to exist, even in the one-dimensional case and even if the solution may be approximated up to a factor of 1.399.
Handling outliers

Physical sensors, such as GPS, may generate inaccurate measurements, which we refer to as outliers.
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- The Fréchet distance and the discrete Fréchet distance are sensitive to outliers.
Handling outliers

How to reduce sensitivity to outliers?

\[ \delta \]
Handling outliers

How to reduce sensitivity to outliers?
Take **shortcuts**!

- Allow the $A$-frog or the $B$-frog (or both) to “ignore” subcurves or points which might be considered as noise.
The one-sided discrete Fréchet distance with shortcuts

\[ A = (a_1, \ldots, a_n), \quad B = (b_1, \ldots, b_m), \] \text{a leash of length } \delta.

Only the } A\text{-frog can skip points.
The one-sided discrete Fréchet distance with shortcuts

\( A = (a_1, ..., a_n) \), \( B = (b_1, ..., b_m) \), a leash of length \( \delta \).

Only the \( A \)-frog can skip points.

- The **one-sided discrete Fréchet distance with shortcuts** is the minimum such \( \delta \) that allows the frogs to reach \( a_m \) and \( b_n \).
The weak discrete Fréchet distance

\( A = (a_1, \ldots, a_n), \ B = (b_1, \ldots, b_m) \), a leash of length \( \delta \).

The frogs are also allowed to jump (one step) backwards.
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- The **weak discrete Fréchet distance** is the **minimum** such $\delta$ that allows the frogs to reach $a_m$ and $b_n$. 
Fréchet under translation

The input curves are not necessarily \textit{aligned}, and one of them needs to be \textit{adjusted}.
Fréchet under translation

The input curves are not necessarily aligned, and one of them needs to be adjusted.

Given $A = (a_1, \ldots, a_n)$, $B = (b_1, \ldots, b_m)$, find a translation $t$ that minimizes the discrete Fréchet distance between $A$ and $B + t$. 

$\delta$
Related work

**Continuous** Fréchet distance under translation:

- **Alt et al. (’01):** $O(m^3 n^3 (m + n)^2 \log (m + n))$-time algorithm for points in 2D, and an algorithm computing a $(1 + \epsilon)$-approximation in $O(\epsilon^{-2} mn)$ time.

- **Wenk (’03):** $O((m + n)^{11} \log (m + n))$-time algorithm for points in 3D (general results for $d$ dimensions and other families of transformations).
Related work

**Discrete** Fréchet distance under translation:

- **Mosig et al. (’05):** $O(m^2 n^2)$-time approximation algorithm for DFD under translation, rotation and scaling in 2D, with approximation factor close to 2.

- **Jiang et al. (’08):** $O(m^3 n^3 \log(m + n))$-time algorithm for DFD under translation, and an $O(m^4 n^4 \log(m + n))$-time algorithm when both rotations and translations are allowed.

- **Ben Avraham et al. (’15):** $O(m^3 n^2 (1 + \log(\frac{n}{m})) \log(m + n))$-time algorithm, based on a dynamic data structure which supports updates and reachability queries in $O(m(1 + \log(n/m))$ time (up next).
The positions graph

\[ G = G(V = A \times B, E = E_A \cup E_B \cup E_{AB}) \]

- \( V \): all possible positions of the frogs.
- \( E \): all possible moves between positions.

\[ E_A = \{\langle (a_i, b_j), (a_{i+1}, b_j) \rangle \} \]
\[ E_B = \{\langle (a_i, b_j), (a_i, b_{j+1}) \rangle \} \]
\[ E_{AB} = \{\langle (a_i, b_j), (a_{i+1}, b_{j+1}) \rangle \} \]
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Sequence of moves (with unlimited leash length) \( \iff \) A path in \( G \) from \((a_1, b_1)\) to \((a_n, b_m)\).
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Introduction

Preliminaries

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The indicator function

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$(a_i, b_j)$ is a **reachable position** (w.r.t. $\sigma$), if there exists a path $P$ in $G$ from $(a_1, b_1)$ to $(a_i, b_j)$, consisting of only valid positions.
Discrete Fréchet distance

Let $d(a_i, b_j)$ denote the **Euclidean distance** between $a_i$ and $b_j$.

**Our indicator function:** given a distance $\delta \geq 0$,

$$
\sigma_\delta(a_i, b_j) = \begin{cases} 
1, & d(a_i, b_j) \leq \delta \\
0, & \text{otherwise}
\end{cases}
$$

The **discrete Fréchet distance** $d_{dF}(A, B)$ is the smallest $\delta \geq 0$ for which $(a_n, b_m)$ is a reachable position w.r.t. $\sigma_\delta$. 
(\(a_i, b_j\)) is an **s-reachable position** (w.r.t. \(\sigma\)), if there exists a path \(P\) in \(G\) from \((a_1, b_1)\) to \((a_i, b_j)\), such that:

- \(\sigma(a_1, b_1) = 1\) and \(\sigma(a_i, b_j) = 1\).
- For each \(b_l, 1 < l < j\), there exists a position \((a_k, b_l) \in P\) that is valid (i.e., \(\sigma(a_k, b_l) = 1\)).
One-sided shortcuts

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(a_i, b_j) is an **s-reachable position** (w.r.t. \( \sigma \)), if there exists a path \( P \) in \( G \) from \((a_1, b_1)\) to \((a_i, b_j)\), such that:

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The **discrete Fréchet distance with shortcuts** \( d_{dF}^s(A, B) \) is the smallest \( \delta \geq 0 \) for which \((a_n, b_m)\) is an s-reachable position w.r.t. \( \sigma_\delta \).
Weak Fréchet distance

Backtracking is allowed!
Remove the directions from \( G \), a new graph: \( G_w = G(V = A \times B, E_w) \)

\[
E_w = \{(u, v) | \langle u, v \rangle \in E_A \cup E_B \cup E_{AB}\}
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Weak Fréchet distance

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$(a_i, b_j)$ is a **w-reachable position** (w.r.t. $\sigma$), if there exists a path $P$ in $G_w$ from $(a_1, b_1)$ to $(a_i, b_j)$ consisting of only valid positions.
Weak Fréchet distance

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The **weak discrete Fréchet distance** $d_{dF}^w(A, B)$ is the smallest $\delta \geq 0$ for which $(a_n, b_m)$ is a w-reaching position w.r.t. $\sigma_\delta$. 
The translation problem

Given two sequences of points \( A = (a_1, \ldots, a_n) \) and \( B = (b_1, \ldots, b_m) \), find a translation \( t^* \) that minimizes \( d_{dF}(A, B + t) \) over all translations \( t \).

Denote:
\[
\hat{d}_{dF}(A, B) = \min_t \{ d_{dF}(A, B + t) \}
\]
\[
\hat{d}^s_{dF}(A, B) = \min_t \{ d^s_{dF}(A, B + t) \}
\]
\[
\hat{d}^w_{dF}(A, B) = \min_t \{ d^w_{dF}(A, B + t) \}
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The algorithm of Ben Avaraham, Kaplan, and Sharir

- **Ben Avraham et al. ('15):** $O(m^3 n^2 (1 + \log(n/m)) \log(m + n))$-time algorithm for the discrete Fréchet distance under translation, based on a dynamic data structure which supports updates and reachability queries in $O(m(1 + \log(n/m))$ time.
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**STEP 1:** Build a **dynamic data structure** for (discrete Fréchet) reachability queries.
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Given sequences $A$ and $B$ and an indicator function $\sigma$: 

The dynamic data structure is constructed in $O(mn)$ time.

- Allows the following operations in $O(m(1 + \log(n/m)))$ time:
  - Reachability query: return TRUE if and only if $(a_n, b_m)$ is a reachable position w.r.t. $\sigma$.
  - A single change in $\sigma$: switch $\sigma(a_i, b_j)$ from 1 to 0 or vice versa.
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**STEP 2:** The **decision problem**: given a distance $\delta$, is there a translation $t$ such that $d_{dF}(A, B + t) \leq \delta$?
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- Given $a \in A$, $b \in B$, consider the **disk** $D_\delta(a - b)$ of radius $\delta$ centered at $a - b$: 

![Disk Diagram](image-url)
STEP 2: The decision problem: given a distance $\delta$, is there a translation $t$ such that $d_{dF}(A, B + t) \leq \delta$?

- Given $a \in A$, $b \in B$, consider the disk $D_\delta(a - b)$ of radius $\delta$ centered at $a - b$:

  $t \in D_\delta(a - b) \iff d(a - b, t) \leq \delta \iff d(a, b + t) \leq \delta$. 

![Diagram showing a circle with radius $\delta$ centered at $a - b$.]
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- Construct the **arrangement** $A_\delta$ of the disks in \{ $D_\delta(a - b)$ | $(a, b) \in A \times B$ \} — it has $O(m^2 n^2)$ cells.

![Diagram of disks and points](image-url)
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- Construct the **arrangement** $A_\delta$ of the disks in $\{D_\delta(a - b) \mid (a, b) \in A \times B\}$ — it has $O(m^2 n^2)$ cells.

- Initialize a **dynamic data structure** for (discrete Fréchet) reachability queries.
**STEP 2:** The **decision problem**: given a distance \( \delta \), is there a translation \( t \) such that \( d_{dF}(A, B + t) \leq \delta \)?

- **Given** \( a \in A, b \in B \), **consider the disk** \( D_\delta(a - b) \) of radius \( \delta \) centered at \( a - b \):
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  t \in D_\delta(a - b) \iff d(a - b, t) \leq \delta \iff d(a, b + t) \leq \delta.
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- **Construct the arrangement** \( A_\delta \) of the disks in \( \{D_\delta(a - b) \mid (a, b) \in A \times B\} \) — it has \( O(m^2n^2) \) cells.

- **Initialize a dynamic data structure** for (discrete Fréchet) reachability queries.

- **Traverse the cells of** \( A_\delta \): when moving between neighboring cells, the data structure is updated and queried in \( O(m(1 + \log(n/m))) \) time.
STEP 2: The decision problem: given a distance $\delta$, is there a translation $t$ such that $d_{dF}(A, B + t) \leq \delta$?

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- Construct the arrangement $A_\delta$ of the disks in $\{D_\delta(a - b) \mid (a, b) \in A \times B\}$ — it has $O(m^2n^2)$ cells.

- Initialize a dynamic data structure for (discrete Fréchet) reachability queries.

- Traverse the cells of $A_\delta$: when moving between neighboring cells, the data structure is updated and queried in $O(m(1 + \log(n/m))$ time.

STEP 3: Use parametric search in order to find an optimal translation (adds only a $O(\log(m + n))$ factor to the running time).
Other variants of DFD

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  - **Eppstein et al. ('92)**: updates and reachability queries in an undirected planar graph in $O(\log |V|)$ time per operation.
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- **DFDS**: reachability by *shortcut paths* — an s-path consists of both valid and non-valid positions, and not every path in $G$ is an s-path.
Other variants of DFD

An efficient dynamic data structure for other variants of DFD?

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- **DFDS**: reachability by **shortcut paths** — an s-path consists of both valid and non-valid positions, and not every path in $G$ is an s-path.

Build a graph of s-paths!
(partial) s-paths

A path $P$ in $G_\delta$ from $(a_i, b_j)$ to $(a_{i'}, b_{j'})$, $i \leq i'$, $j \leq j'$, is a partial s-path, if for each $b_l$, $j \leq l < j'$, there exists a position $(a_k, b_l) \in P$ that is valid (i.e., $\sigma_\delta(a_k, b_l) = 1$).
(partial) s-paths

A path $P$ in $G_\delta$ from $(a_i, b_j)$ to $(a_{i'}, b_{j'})$, $i \leq i'$, $j \leq j'$, is a **partial s-path**, if for each $b_l$, $j \leq l < j'$, there exists a position $(a_k, b_l) \in P$ that is valid (i.e., $\sigma_\delta(a_k, b_l) = 1$).

A path $P$ in $G_\delta$ from $(a_i, b_j)$ to $(a_{i'}, b_{j'})$, $i \leq i'$, $j \leq j'$, is an **s-path**, if it is a partial s-path and also $\sigma_\delta(a_i, b_j) = \sigma_\delta(a_{i'}, b_{j'}) = 1$. 
The graph of (partial) s-paths

Consider $G_\delta = G(V = A \times B, E = E'_A \cup E'_B)$

$$E'_A = \{ \langle (a_i, b_j), (a_{i+1}, b_j) \rangle \mid \sigma_\delta(a_i, b_j) = 0 \}$$

$$E'_B = \{ \langle (a_i, b_j), (a_i, b_{j+1}) \rangle \mid \sigma_\delta(a_i, b_j) = 1 \}$$
Properties of $G_\delta$

**Property 1**

All the paths in $G_\delta$ are partial s-paths.

**Property 2**

$G_\delta$ is a set of rooted binary trees (where the root is a vertex of out-degree 0).
Lemma

\((a_n, b_m)\) is an \(s\)-reachable position in \(G\) w.r.t. \(\sigma_\delta\), if and only if

- \(\sigma_\delta(a_1, b_1) = 1\) and \(\sigma_\delta(a_n, b_m) = 1\).
- The root of \((a_1, b_1)\) in \(G_\delta\) is \((a_i, b_m)\), for some \(1 \leq i \leq n\).
Lemma

\[(a_n, b_m)\] is an s-reachable position in \(G\) w.r.t. \(\sigma_\delta\), if and only if

- \(\sigma_\delta(a_1, b_1) = 1\) and \(\sigma_\delta(a_n, b_m) = 1\).
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Proof.

\(\iff\): By Property 1.
Lemma

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\[
\begin{align*}
\sigma_\delta(a_1, b_1) &= 1 \quad \text{and} \quad \sigma_\delta(a_n, b_m) = 1. \\
The \textbf{root} of \((a_1, b_1)\) in \(G_\delta\) is \((a_i, b_m)\), for some \(1 \leq i \leq n\).
\end{align*}
\]

Proof.

\(\iff\): By Property 1.

\(\Rightarrow\): Clearly, \(\sigma_\delta(a_1, b_1) = 1\) and \(\sigma_\delta(a_n, b_m) = 1\),

Let \(P\) be an s-path in \(G\) from \((a_1, b_1)\) to \((a_n, b_m)\).

Let \(P'\) be the path in \(G_\delta\) from \((a_1, b_1)\) to its root.

\(P'\) is always not above \(P\): if a position \((a_i, b_j)\) is an s-reachable position in \(G\), then there exists a position \((a_{i'}, b_j) \in P', i' \leq i\), such that \(\sigma_\delta(a_{i'}, b_j) = 1\).
Lemma

\((a_n, b_m)\) is an s-reachable position in \(G\) w.r.t. \(\sigma_\delta\), if and only if

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Proof.

![Diagram showing the proof of the lemma](image)
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Proof.
A Link-Cut tree

We represent $G_\delta$ using the Link-Cut tree data structure:

*Sleator and Tarjan ('83)*

The Link-Cut tree data structure stores a set of rooted trees and supports the following operations in $O(\log n)$ amortized time:

- $Link(v, u)$ — connect a root $v$ to another node $u$ as its child.
- $Cut(v)$ — disconnect the subtree rooted at $v$ from its tree.
- $FindRoot(v)$ — find the root of the tree to which $v$ belongs.
The dynamic data structure for DFDS

- Switch $\sigma_\delta(a_i, b_j)$ from 1 to 0:
  - remove the edge $\langle (a_i, b_j), (a_i, b_{j+1}) \rangle$ ($\text{Cut}(a_i, b_j)$).
  - add the edge $\langle (a_i, b_j), (a_{i+1}, b_j) \rangle$ ($\text{Link}((a_i, b_j), (a_{i+1}, b_j))$).

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- Reachability query: return TRUE if and only if
  (i) $\sigma_\delta(a_1, b_1) = \sigma_\delta(a_n, b_m) = 1$, and
  (ii) $\text{FindRoot}(a_1, b_1)$ is $(a_i, b_m)$ for some $1 \leq i \leq n$. 
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**Theorem**

Given sequences $A$ and $B$ with $n$ and $m$ points respectively in the plane, $d_{dF}^s(A, B)$ can be computed in $O(m^2 n^2 \log^2(m + n))$-time.
Back to the algorithm of BKS

Assume the points of $A$ and $B$ are in $\mathbb{R}^d$: 
Back to the algorithm of BKS

Assume the points of $A$ and $B$ are in $\mathbb{R}^d$:

- The size of the arrangement of balls, $A_\delta$, changes to $O(m^d n^d)$. 
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Back to the algorithm of BKS

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- DFDS: $O(m^d n^d \log^2(m + n))$ (our data structure)
- WDFD: $O(m^d n^d \log^2(m + n))$ (the data structure of Eppstein et al.)
A more direct approach for translation in $\mathbb{R}^d$

$A = (a_1, \ldots, a_n), \; B = (b_1, \ldots, b_m)$ — points in $\mathbb{R}^d$.

$S(o, r)$ - the sphere with center $o$ and radius $r$. 
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**New indicator function:**

$$\sigma_{S(o,r)}(a_i, b_j) = \begin{cases} 
1, & d(a_i - b_j, o) \leq r \\
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**Lemma**

Let $S = S(t^*, \delta)$ be a smallest sphere for which $(a_n, b_m)$ is a reachable position w.r.t. $\sigma_S$. Then, $\widehat{d_{dF}}(A, B) = d_{dF}(A, B + t^*) = \delta$. 

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- $S$ is the smallest sphere for which $(a_n, b_m)$ is a reachable position w.r.t. $\sigma_S \Rightarrow \delta' \geq \delta$. 


New goal: find the smallest sphere $S$ for which $(a_n, b_m)$ is a reachable position w.r.t. $\sigma_S$. 
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  - DFDS, WDFD: $O(mn \log(m + n))$
Translation in 1D

**New indicator function**: given a range \([s, t]\),

\[
\sigma_{[s,t]}(a, b) = \begin{cases} 
1, & s \leq a - b \leq t \\ 
0, & \text{otherwise} 
\end{cases}
\]
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Let \(\mathcal{D} = \{a_i - b_j \mid a_i \in A, b_j \in B\}\).

**New goal:** Find the smallest feasible range delimited by two points of \(\mathcal{D}\).
Algorithm:

- Sort the values in $\mathcal{D} = \{d_1, \ldots, d_l\}$ such that $d_1 < d_2 < \cdots < d_l$, where $l = mn$.
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- Set $p \leftarrow 1, q \leftarrow 1$.

$[d_p, d_q]$ is not a feasible range

<table>
<thead>
<tr>
<th>$d_1$</th>
<th>$d_2$</th>
<th>$d_3$</th>
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$p \ q$
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$\hspace{1cm}$

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\[
\begin{array}{cccccccccccc}
  & d_1 & d_2 & d_3 & d_4 & & & & & & & d_l \\
  p & & & & & & & & & & & \\
  q & & & & & & & & & & & \\
\end{array}
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- Set $p \leftarrow 1$, $q \leftarrow 1$.
- While $q \leq l$, if $(a_n, b_m)$ is a reachable position in $G$ w.r.t. $\sigma_{[d_p, d_q]}$, set $p \leftarrow p + 1$, else set $q \leftarrow q + 1$.
- Return the translation corresponding to the smallest feasible range $[d_p, d_q]$ that was found during the while loop.

$[d_p, d_q]$ is not a feasible range

\[
\begin{array}{cccccccccccc}
  d_1 & d_2 & d_3 & d_4 & & & & & & & & d_l \\
\end{array}
\]

\[\uparrow \quad \uparrow \]

$p \quad q$
Algorithm:

- Sort the values in $\mathcal{D} = \{d_1, \ldots, d_l\}$ such that $d_1 < d_2 < \cdots < d_l$, where $l = mn$.
- Set $p \leftarrow 1$, $q \leftarrow 1$.
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Theorem

Let $A$ and $B$ be two sequences of $n$ and $m$ points ($m \leq n$), respectively, on the line. Then, $\hat{d}_{dF}(A, B)$ can be computed in $O(m^2 n (1 + \log(n/m))$ time, and $\hat{d}_{dF}^s(A, B)$ and $\hat{d}_{dF}^w(A, B)$ can be computed in $O(mn \log(m + n))$ time.
Balanced Optimization Problem (on Graphs)

\[ G = (V, E, w) \] – a \textbf{weighted graph} with \( n \) vertices and \( m \) edges.

\( \mathcal{F} \) – a set of \textbf{feasible subsets} of \( E \).
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For \( S \subseteq E \), let \( S_{max} = \max\{w(e) : e \in S\} \) and \( S_{min} = \min\{w(e) : e \in S\} \).
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\textbf{Definition: BOP}

Find a feasible subset \( S^* \in \mathcal{F} \) which minimizes \( S_{\text{max}} - S_{\text{min}} \) over all \( S \in \mathcal{F} \).
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**Definition: BOP**

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Goal: find the smallest feasible range.
Martello et al. ('84): a general optimization algorithm BOP:

A general scheme for BOP

Definitions

Given a feasibility decider that decides whether a subset is feasible or not in $f(l)$ time, their algorithm finds an optimal range in $O(lf(l) + l \log l)$-time.

Especially useful when an efficient dynamic version of the feasibility decider is available.

We present an alternative scheme for BOP - does not require a dynamic version of the feasibility decider.

$[d_p, d_q]$ is not a feasible range

$p q$
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尤其是有用的，特别是当有效率的动态版本的可行性判断器可用时。

我们提出一个BOP的替代方案—不需要动态版本的可行性判断器。
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$d_1 \; d_2 \; d_3 \; d_4 \; \ldots \; \ldots \; \ldots \; \ldots \; d_l$

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The matrix of ranges

- Let $w_1 = w(e_1) < w_2 = w(e_2) < \cdots < w_m = w(e_m)$.
- Let $M$ be the matrix whose rows and columns correspond to $w_1, w_2, \ldots, w_m$.
- Cell $M_{i,j} \equiv \text{Range } [w_i, w_j]$. 

![Matrix of ranges diagram]
$M_{i,j}$ is contained in all the ranges $M_{i',j'}$ with $i' \leq i \leq j \leq j'$.
- $M_{i,j}$ is contained in all the ranges $M_{i',j'}$ with $i' \leq i \leq j \leq j'$.
- Perform a binary search in row $\frac{m}{2}$ to find the smallest feasible range $M_{\frac{m}{2},j} = [\frac{w_m}{2}, w_j]$ in this row.
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- $M_{\frac{m}{2},j}$ induces a partition of $M$ into 4 submatrices: $M_1, M_2, M_3, M_4$. 
None of the ranges in $M_1$ is a feasible range.
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Each of the ranges in $M_4$ is at least as large as $M_{\frac{m}{2}j}$.

We may ignore $M_1$ and $M_4$ and continue recursively with the submatrices $M_2$ and $M_3$. 

![Matrix of ranges diagram]
None of the ranges in $M_1$ is a feasible range.

Each of the ranges in $M_4$ is at least as large as $M_{\frac{m}{2}, j}$.

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The algorithm

Each recursive call is associated with:

- a submatrix $M' = M([p, p'] \times [q', q])$ of $M$.
- a corresponding graph $G' = G([p, p'] \times [q', q])$.

A recursive call has 2 steps:

1. **Perform a binary search in the middle row** of $M'$ to find the smallest feasible range in this row, using the corresponding graph $G'$.
2. **Construct two new graphs** for the two submatrices of $M'$ in which we still need to search in the next level of the recursion.
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The scheme requires the following properties of $G'$:

1. The size of $G'$ is $O(|M'|)$.
2. Given $G'$, the feasibility decider can answer a feasibility query for any range in $M'$, in $O(f(|G'|))$ time.
3. Constructing the graphs for the next level takes $O(|G'|)$ time.
Running time

The recursion tree consists of $O(\log m)$ levels.

- The $i$’th level is associated with $2^i$ disjoint submatrices of $M$.
- In the $i$’th level we apply the recursive algorithm to each of the $2^i$ submatrices associated with this level.
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Let $\{M^i_k\}_{k=1}^{2^i}$ be the submatrices associated with the $i$’th level.
Let $G^i_k$ be the graph corresponding to $M^i_k$. 
In each recursive call:

1. The size of $G^i_k$ is $O(M^i_k)$.
2. The feasibility decider runs in $O(f(|G^i_k|))$ time $\Rightarrow$ the binary search in $M^i_k$ runs in $O(f(|M^i_k|) \log |M^i_k|)$ time.
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The total time spent on the $i$'th level is

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Lemma

The total size of the matrices in each level is at most $2m$.

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If $f(|M_k^i|) = O(|M_k^i|)$, then we get $O(m \log^2 n)$. 
The Most Uniform Path problem (MUPP)

\[ G = (V, E, w) \] is a weighted graph with \( n \) vertices and \( m \) edges.

**Definition: MUPP**

Given two vertices \( s, t \in V \), find a path \( P^* \) in \( G \) between \( s \) and \( t \), which minimizes \( P_{\text{max}} - P_{\text{min}} \), over all paths \( P \) between \( s \) and \( t \).
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- Our scheme: simpler algorithm with the same running time.
Applying our BOP scheme:

- A feasible subset $S \in \mathcal{F}$ is a path in $G$ between $s$ and $t$. 
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- In a **recursive call**: Let $M'$ be the submatrix and $G'$ the graph associated with it. Maintain the following properties of $G'$:
  1. The size of $G'$ is at most $O(|M'|)$.
  2. Given a range $[w_p, w_q]$ in $M'$, there exists a path between $s$ and $t$ in $G'$ with edges in the range $[w_p, w_q]$ if and only if such a path exists in $G'$.
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- The **feasibility decider**: a BFS algorithm (which ignores edges outside the given range) runs in $O(|G'|)$ time.
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- The feasibility decider: a BFS algorithm (which ignores edges outside the given range) runs in \( O(|G'|) \) time.
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- The **feasibility decider**: a BFS algorithm (which ignores edges outside the given range) runs in $O(|G'|)$ time.
Construction of the graph $G'' = G([p, p'] \times [q', q])$, given $G'$:

1. Remove from $G'$ all the edges $e$ such that $w(e) \notin [w_p, w_q]$.
2. Contract edges with weights in the range $(w_{p'}, w_{q'})$.
3. Remove all the isolated vertices.

$$G' = G([2, 7] \times [10, 15]) \Rightarrow G'' = G([3, 5] \times [11, 13])$$
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Construction of the graph $G'' = G([p, p'] \times [q', q])$, given $G'$:

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Running time: \( O(|G'|) \).
WDFD under translation in 1D can be viewed as BOP:

\[ [s, t] \text{ is a feasible range } \iff (a_n, b_m) \text{ is a w-reaching position in } G_w \text{ w.r.t. } \sigma_{[s,t]} \cdot \]
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WDFD under translation in 1D is a special case of MUPP!
MUPP $\Rightarrow$ WDFD under translation in 1D

For MUPP we need a weighted graph $\tilde{G}_w = (\tilde{V}_w, \tilde{E}_w, \omega)$

$\tilde{V}_w = (A \times B) \cup \{v_e | e \in E_w\}$,

$\tilde{E}_w = \{(u, v_e), (v_e, v) | e = (u, v) \in E_w\}$,

and $\omega(((a_i, b_j), v_e)) = a_i - b_j$.

\[ \begin{array}{c|c|c|c|c|c} 
 a_1 & a_2 & a_3 & a_4 & a_5 \\
 b_1 & \bullet & \bullet & \bullet & \bullet & \bullet \\
 b_2 & \bullet & \bullet & \bullet & \bullet & \bullet \\
 b_3 & \bullet & \bullet & \bullet & \bullet & \bullet \\
 b_4 & \bullet & \bullet & \bullet & \bullet & \bullet \\
 b_5 & \bullet & \bullet & \bullet & \bullet & \bullet \\
\end{array} \]

$\Rightarrow$

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$\tilde{G}_w$
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$(a_n, b_m)$ is a **w-reachable position** in $G_w$ w.r.t. $\sigma_{[s,t]}$ $\iff$ there exists a path $\tilde{P}$ between $(a_1, b_1)$ and $(a_n, b_m)$ in $\tilde{G}_w$ such that for each edge $e \in \tilde{P}$, $\omega(e) \in [s, t]$. 
**MUPP ⇒ WDFD under translation in 1D**

MUPP between \((a_1, b_1)\) and \((a_n, b_m)\) in \(\tilde{G}_w\) ⇒ WDFD under translation in 1D.

**Theorem**

Let \(A = (a_1, \ldots, a_n)\) and \(B = (b_1, \ldots, b_m)\) be two sequences of points in 1D. Then, the weak discrete Fréchet distance under translation, \(\tilde{d}_{dF}^w(A, B)\), can be computed in \(O(mn\log^2(m + n))\) time.
The Most Uniform Spanning Tree problem (MUTP)

\[ G = (V, E, w) \] is a weighted graph with \( n \) vertices and \( m \) edges.

**Definition: MUTP**

Find a spanning tree \( T^* \) of \( G \), which minimizes \( T_{max} - T_{min} \), over all spanning trees \( T \) of \( G \).
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**Our algorithm**: slower by a factor of \( \log n \), BUT does not require any special data structures, and has easy and shorter description.
A new variant: The discrete Fréchet gap

Two frogs, two curves: $A = (a_1, \ldots, a_n)$, $B = (b_1, \ldots, b_n)$.
Same rules: traverse all the points in order, no backtracking.

New indicator function: given a range $[s, t]$,

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\sigma_{[s,t]}(a, b) = \begin{cases} 
  1, & s \leq d(a, b) \leq t \\
  0, & \text{otherwise}
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0, & \text{otherwise}
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The discrete Fréchet gap $(d^g_{dF}(A, B))$ is the smallest range $[s, t]$, $s \geq t \geq 0$, for which $(a_n, b_m)$ is a reachable position w.r.t. $\sigma_{[s,t]}$. 
Discuss

Is this a better variant?

- Gives a better reflection of reality?
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- Gives a better reflection of reality? *sometimes, but not always.*
Is this a better variant?

- Gives a better reflection of reality? sometimes, but not always.
- Handling outliers? no.
Let us combine gap and shortcuts...

\[ A = (a_1, ..., a_n), \quad B = (b_1, ..., b_n), \] a leash of length \( \delta \).

Only the \( A \)-frog can skip points.

- The (one-sided) discrete Fréchet gap with shortcuts is the smallest range \([s, t]\), \( s \geq t \geq 0 \), for which \((a_n, b_m)\) is an s-reachable position w.r.t. \( \sigma_{[s,t]} \).
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That seems to give better results!
What is between Gap and Translation?

Is there some connection between the discrete Fréchet gap and the discrete Fréchet distance under translation?
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Is there some connection between the discrete Fréchet gap and the discrete Fréchet distance under translation?

DFDS and WDFD, both in 1D under translation, are in some sense analogous to their respective gap variants (in $d$ dimensions and no translation):

- We can use similar algorithms to compute them, but with different indicator functions!
Thank You!